INTEGRABLE SYSTEMS AND SYMPLECTIC GEOMETRY

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Abstract.

1. Introduction

Recommended exercises are marked by *(*)*.

Difficult exercises are marked by *(!)*.

Text appearing in light grey has NOT been discussed in the lecture and is therefore not exam material.

2. Mechanics and symplectic geometry

2.1. **Hamilton's equations.** Here is Newton's equation of motion from classical mechanics,

$$\ddot{q}(t) = -\nabla V(q) \tag{1}$$

describing the position depending on time $t \mapsto q(t) \in \mathbb{R}^n$ of a particle of unit mass submitted to a force field of a potential $V : \mathbb{R}^n \to \mathbb{R}$. Here we take the potential to not depend on time, nor on the velocity $\dot{q}(t)$. This is an second order ODE in $\mathbb{R}^n$.

Hamilton's reformulation of that equation goes as follows: Set $p(t) = \dot{q}(t) \in \mathbb{R}^n$ and view it as another variable of the system, called *momentum*. This yields the equation of motion $\dot{p}(t) = -\nabla V(q)$ coupled with $\dot{q}(t) = p(t)$ Setting

$$H(q,p) = \frac{\|p\|^2}{2} + V(q) \tag{2}$$

for Euclidean norm $\|p\|^2 = p_1^2 + \cdots + p_n^2$ allow us to write Newton's equation as a system of coupled first order ODEs

$$\dot{q}_i(t) = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i(t) = -\frac{\partial H}{\partial q_i}. \tag{3}$$

These are called *Hamilton's equations* and H is called *Hamiltonian (function)*. Physically speaking, H measures the total energy of the system, and is composed of the *kinetic energy*

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$\frac{\|p\|^2}{2}$ and the *potential energy* $V(q)$. It is easy to check that the solutions preserve energy, meaning that the function $t \mapsto H(q(t), p(t))$ is constant in t,

$$\frac{dH}{dt} = \sum_{i=1}^{n} \frac{\partial H}{\partial q_i} \dot{p}_i + \frac{\partial H}{\partial q_i} \dot{q}_i = 0,$$

where we have used (3).

Example 2.1 (Harmonic oscillator). Let $n = 1$ and $V(q) = \alpha q^2$ for some $\alpha > 0$. Then (3) describe the motion of a unit mass attached to a spring which is at rest at $q = 0$. The parameter α has something to do with the stiffness of the spring. To say something about the harmonic oscillator, note that the total energy is preserved, i.e. the function $t \mapsto H(q(t), p(t))$ is constant for all curves solving (3). This allows us to say something very strong about the solutions of these equations, namely that they are constrained to the ellipses $\{\frac{p^2}{2} + \alpha q^2 = c\}$. We can draw the so-called *phase portrait* of the Harmonic oscillator, see **??**. Of course, we could just have written down explicit solutions to the Harmonic oscillator instead, but closed formulae for this kind of problem exist only in very rare cases. The kind of geometric arguments above will inspire much of the rest of this lecture.

2.2. **Symplectic geometry.** The natural arena to set up a problem of classical mechanics is a *symplectic manifold*.

Definition 2.2. *A* symplectic manifold *is a pair* (X^{2n}, ω) *consisting of a smooth even-dimensional manifold* X *and a differential two form which is closed and non-degenerate.*

By *closed*, we mean that its exterior differential vanishes, i.e. $d\omega = 0$. By *non-degenerate* we mean the assignment $Z \mapsto \iota(Z)\omega = \omega(Z, -)$ defines a bundle isomorphism $TX \to T^*X$. Non-degeneracy allows us to turn any smooth function $H \in C^\infty(X)$ into a vector field X_H by setting

$$dH = \iota(X_H)\omega. \tag{4}$$

Definition 2.3. *In this context, we call* $H \in C^\infty(X)$ *a* Hamiltonian (function)*, we call* $X_H \in \Gamma(TX)$ Hamiltonian vector field*, and we call the flow* $t \mapsto \phi_t^H$ *it generates* Hamiltonian flow*. The triple* (X, ω, H) *is called* Hamiltonian system*.*

By *the flow it generates* we mean the one-parametric family of diffeomorphisms ϕ_t^H satisfying $\frac{d\phi_t^H}{dt} = X_H$ for all times t. **All throughout, we assume that Hamiltonian flows are defined for all times.** The Hamiltonian flow of H preserves H and the symplectic form ω,

$$H \circ \phi_t^H = H, \quad (\phi_t^H)^*\omega = \omega. \tag{5}$$

Indeed, differentiating with respect to time, we obtain, in the first case

$$\frac{d}{dt}(H \circ \phi_t^H) = dH(X_H) = (\iota(X_H)\omega)(X_H) = \omega(X_H, X_H) = 0.$$

The second equality is (4) and the last one follows from the skew-symmetry $\omega(X,Y) = -\omega(Y,X)$ of differential two forms. In the second case we obtain

$$\frac{d}{dt}(\phi_t^H)^*\omega = \mathcal{L}_{X_H}\omega = d\iota(X_H)\omega + \iota(X_H)d\omega = ddH + 0 = 0,$$

where $\mathcal{L}_Y$ is the *Lie derivative* of tensor fields along a vector field Y. In case the tensor field is a differential form $\alpha \in \Omega^1(X)$, it can be computed by the co-called *Cartan's magic formula* $\mathcal{L}_Y\alpha = d\iota(Y)\alpha + \iota(Y)d\alpha$, which is what we have used to find the second equality.

Example 2.4. Let $\mathbb{R}^2 = \{(q,p)\}$ be equipped with the differential two form

$$\omega_0 = dq \wedge dp.$$

This is just the standard area form. It is closed, $d\omega_0 = ddx \wedge dy + dx \wedge ddy = 0$ (every differential form in the top degree is automatically closed anyways) and furthermore we have

$$\iota(\partial_q)\omega_0 = dp, \quad \iota(\partial_p)\omega_0 = -\iota(\partial_p)dp \wedge dq = -dq \tag{6}$$

where $\partial_q, \partial_p \in T\mathbb{R}^2$ denote the vector fields pointing in the coordinate directions. This implies non-degeneracy, since ∂_q, ∂_p span $T\mathbb{R}^2$ and dq, dp span $T^*\mathbb{R}^2$. Let us now look at some Hamiltonians:

$$H_1(q,p) = p, \quad H_2(q,p) = q^2 + p^2.$$

(4) yields

$$X_{H_1} = \partial_q, \quad X_{H_2} = 2p\partial_q - 2q\partial_q.$$

Indeed, (6) shows that we can go from dH to X_H by replacing every dp by ∂_q and every dq by $-\partial_p$ The first Hamiltonian flow acts by translation in the q-direction and the second one by rotation.

The product of two symplectic manifolds is again symplectic.

Exercise 2.5. Let $\mathbb{R}^4 = \{(q_1,p_1,q_2,p_2)\}$ be equipped with $\omega_0 = dq_1 \wedge dp_1 + dq_2 \wedge dp_2$. The Hamiltonian

$$G(q_1,p_1,q_2,p_2) = p_1q_2 - p_2q_1$$

is sometimes called *angular momentum.* Compute its flow ϕ_t^G. How does this flow act on the (q_1,q_2)- and on the (p_1,p_2)-planes?

For any n, denote the coordinates on $\mathbb{R}^{2n}$ by $q_1,\ldots,q_n,p_1,\ldots,p_n$ and let

$$\omega_0 = \textstyle\sum_{i=1}^n dq_i \wedge dp_i.$$

This is called the *standard symplectic form.* Let H be a Hamiltonian. It differential is given by

$$dH = \sum_{i=1}^n \left(\frac{\partial H}{\partial q_i}dq_i + \frac{\partial H}{\partial p_i}dp_i\right),$$

and thus

$$X_H = \sum_{i=1}^{n} \left(\frac{\partial H}{\partial p_i} \partial_{q_i} - \frac{\partial H}{\partial q_i} \partial_{p_i} \right)$$

In other words, the system $(\dot{q}(t), \dot{p}(t)) = X_H$ exactly corresponds to Hamilton's equations (3)! This shows that we have generalized them to manifolds.

Definition 2.6. *We call* (4) Hamilton's equation *(singular!) from now on.*

This means that the first equation in (5) is a generalized form of energy conservation. As for the second equation, let us make the following historical remark:

Remark 2.7. Liouville noticed in the 19th century that phase space of systems from classical mechanics carries a natural measure (or volume form) which is given, in so-called canonical coordinates, by

$$\mathrm{vol} = dx_1 \wedge dy_1 \wedge \ldots \wedge dx_n \wedge dy_n,$$

which is preserved by the mechanical system. The fact that the symplectic form ω is preserved is often (for example in [5]) attributed to Arnol'd. It is a strict refinement of Liouville's theorem. Indeed, we can write

$$n! \, \mathrm{vol} = \omega_0^{\wedge n}$$

for the standard symplectic form ω_0. Since $(\phi_t^H)^* \omega_0 = \omega_0$ by (5), we deduce that $(\phi_t^H)^* \mathrm{vol} = \mathrm{vol}$. This refinement has spectacular consequences! Indeed, symplectic topology studies different so-called *symplectic rigidity* phenomena. Those are properties exhibited by symplectic and Hamiltonian flows which volume-preserving flows do not have.

Definition 2.8. *A diffeomorphism* $\psi \in \mathrm{Diff}(X)$ *of a symplectic manifold* (X, ω) *is called* symplectomorphism *if* $\psi^* \omega = \omega$. *The set of these forms a subgroup*

$$\mathrm{Symp}(X, \omega) \subset \mathrm{Diff}(X).$$

As we have shown in (5), every Hamiltonian flow is a symplectomorphism. Furthermore, conjuagting a Hamiltonian flow by a symplectomorphism yields another Hamiltonian flow:

$$\psi \circ \phi_t^H \circ \psi^{-1} = \phi_t^{H \circ \psi^{-1}} \tag{7}$$

for all $H \in C^\infty(X)$ and all $\psi \in \mathrm{Symp}(X, \omega)$. In other words, conjugation by a symplectomorphism corresponds to pulling back its Hamiltonian function. This can be proved by differentiating with respect to t, which yields $\psi_* X_H = X_{H \circ \psi^{-1}}$. This latter equation can be proved by computing

$$\begin{aligned}
\iota(\psi_* X_H)\omega &= \omega(\psi_* X_H, \cdot) \\
&= \omega(\psi_* X_H, \psi_* \psi_*^{-1}(\cdot)) \\
&= \psi^* \omega(X_H, \psi_*^{-1}(\cdot)) \\
&= \omega(X_H, \psi_*^{-1}(\cdot)) \\
&= (\iota(X_H)\omega)(\psi_*^{-1}(\cdot)) \\
&= dH(\psi_*^{-1}(\cdot)) \\
&= d(H \circ \psi^{-1})(\cdot),
\end{aligned}$$

proving the claim by Hamilton's equation. In the fourth line, we have crucially used that ψ is a symplectomorphism, $\psi^*\omega = \omega$.

It is natural to ask now whether Hamiltonian flows are a group, too. This is correct, provided we allow for the Hamiltonians to be time-dependent! A *time-dependent Hamiltonian* is given by a function

$$H\colon [0,1] \times X \to \mathbb{R}, \quad H(t,x) = H_t(x).$$

We can defined X_H^t by $dH_t = \iota(X_H^t)\omega$ for every t. We still denote the corresponding flow by ϕ_t^H.

Definition 2.9. *A diffeomorphism of X is called* Hamiltonian diffeomorphism *if it is the time-one map ϕ_1^H of a (possibly time-dependent) Hamiltonian H.*

Although this is not immediate, it can be shown that these objects form a group, denoted by $\mathrm{Ham}(X,\omega)$. See [5, Section 1.4] for details and proofs.

We have inclusions

$$\mathrm{Ham}(X,\omega) \subset \mathrm{Symp}(X,\omega) \subset \mathrm{Diff}(X). \tag{8}$$

One can wonder how big the differences are between these groups. Let us briefly discuss the case of the inclusion $\mathrm{Ham}(X,\omega) \subset \mathrm{Symp}(X,\omega)$.

Proposition 2.10. *For any Hamiltonian flow ϕ_t^H and any smooth $\gamma\colon S^1 \to X$, we have*

$$\int_{[0,1]\times S^1} C_\gamma^* \omega = 0, \quad \textit{where } C_\gamma\colon [0,1] \times S^1 \to X, \quad C_\gamma(s,t) = \phi_s^H(\gamma(t)). \tag{9}$$

Geometrically, this means that any cylinder swept out by a curve under a Hamiltonian flow has vanishing symplectic area.

Proof. Compute

$$\frac{\partial C_\gamma}{\partial s} = X_s^H(\phi_s^H(\gamma(t))), \quad \frac{\partial C_\gamma}{\partial t} = (\phi_s^H)_* \dot{\gamma}(t),$$

and plug it in

$$\begin{aligned}\int_{[0,1]\times S^1} C_\gamma^*\omega &= \int_{[0,1]}\int_{S^1} \omega(X_s^H(\phi_s^H(\gamma(t))), (\phi_s^H)_*\dot\gamma(t))dtds\\ &= \int_{[0,1]}\int_{S^1} dH_s((\phi_s^H)_*\dot\gamma(t))dtds\\ &= \int_{[0,1]}\int_{S^1} \frac{(H_s\circ\phi_s^H\circ\gamma)(t)}{dt}dtds = 0\end{aligned}$$

In the second line, we have used Hamilton's equation and in the last line the fundamental theorem of calculus combined with the fact that γ is defined on S^1. ☐

Using this, we can construct an example of a non-Hamiltonian symplectomorphism.

Example 2.11. Let $X = S^1\times\mathbb{R}$, where we view $S^1 = \mathbb{R}/\mathbb{Z}$. The symplectic form $\omega_0 = dq\wedge dp$ on $\mathbb{R}^2$ is translation-invariant. Therefore, it descends to a well-defined form ω on X. Note that this form is exact, i.e. it satisfies

$$\omega = d(-pdq). \tag{10}$$

Now let $\psi_r(q,p) = (q, p+r)$ be a translation in the p-direction on X by an amount r. This is a symplectomorphism for all $r\in\mathbb{R}$. However we will see that it is Hamiltonian only if $r = 0$. Define the curve

$$\gamma\colon S^1\to X, \quad \gamma(t) = (t,0).$$

Suppose that ψ_r is a Hamiltonian diffeomorphism, i.e. there is ϕ_t^H with $\psi_r = \phi_t^H$. Then C_γ would have vanishing symplectic area by Proposition 2.10. However, by (10) and Stokes, we can compute

$$\int_{[0,1]\times S^1} C_\gamma^* d(-pdq) = \int_{[0,1]\times S^1} dC_\gamma^*(-pdq) = -\int_{\gamma(S^1)} pdq + \int_{\psi_r(\gamma(S^1))} pdq = r.$$

Since $\gamma(S^1) = S^1\times\{p=0\}$ and $\psi_r(\gamma(S^1)) = S^1\times\{p=r\}$, we find that this is equal to $0+r=r$. This proves that ψ_r is not Hamiltonian whenever $r\neq 0$. Let us point out that it is not enough to use that the cylinder bounding the circles $S^1\times\{p=0\}$ and $S^1\times\{p=r\}$ has non-zero symplectic area! Indeed the Hamiltonian flow ϕ_t^H ending on ψ_r may be much more complicated than just the obvious translation $\phi_t^H = \psi_{tr}$.

Somewhat miraculously, there is a converse to Proposition 2.10, due to Banyaga.

Theorem 2.12. *Let (X,ω) be compact and connected and let $s\mapsto\psi_s\in \mathrm{Symp}(X,\omega)$ be a smooth path of symplectomorphisms starting at $\psi_0 = \mathrm{id}$. If the area (9) (now swept out by the symplectic family $s\mapsto\psi_s$) vanishes for all $\gamma\colon S^1\to X$, then $\psi_1\in\mathrm{Ham}(X,\omega)$.*

Proving this is outside the scope of this lecture. We refer to [4, Chapter 10], and Theorem 10.2.5, in particular. Note that checking this condition on *all maps* $\gamma\colon S^1\to X$ is not convenient. Instead, on can restrict one's attention to homological data to obtain the so-called *flux map* which has values in $H^1(X;\mathbb{R})$. We will encounter its Lagrangian cousin later in this lecture, we refer again to [4] or to [5, Chapter 14] for details.

From the above discussion, it follows in particular that if $H^1(X;\mathbb{R})=0$, then any path of symplectomorphisms ψ_t is automatically Hamiltonian. This fact is much more elementary. To see this, let $Y_t \in \Gamma(TX)$ be the family of vector fields satisfying $\frac{d\psi_t}{dt}=Y_t$. In other words, Y_t is the *velocity vector* (in some infinite-dimensional space...) of the curve $t\mapsto\psi_t$. Since $\psi_t^*\omega=\omega$, we obtain $\psi_t^*(\mathcal{L}_{Y_t}\omega)=0$, which implies, by Cartan's magic formula, that

$$0 = d\iota(Y_t)\omega + \iota(Y_t)d\omega = d\iota(Y_t)\omega, \tag{11}$$

meaning that the one-form $\iota(Y_t)\omega$ is closed for all t. Since $H^1(X;\mathbb{R})=0$, de Rham's theorem implies that every $\iota(Y_t)\omega$ has a primitive function H_t, meaning

$$dH_t = \iota(Y_t)\omega.$$

But this is exactly Hamilton's equation (4). Up to checking that the t dependence in $H_t = H(t,\cdot)$ is smooth (this is actually somewhat subtle), this proves the claim.

Remark 2.13. Let us briefly comment on de Rham's theorem and how it is used here. It says that the *de Rham cohomology group* $H^1_{\mathrm{dR}}(X)$ is isomorphic to the usual cohomology $H^1(X;\mathbb{R})$. However de Rham cohomology precisely measures by how much a closed form fails to be exact, i.e.

$$H^1_{\mathrm{dR}}(X) = \frac{\ker(d\colon \Omega^1(X)\to\Omega^2(X))}{\mathrm{im}(d\colon C^\infty(X)\to\Omega^1(X))} = \frac{\Omega^1_{\mathrm{cl}}(X)}{\Omega^1_{\mathrm{ex}}(X)}.$$

Therefore, every closed one form has a primitive function whenever $H^1(X;\mathbb{R})=0$.

Remark 2.14. It is sometimes useful to view the spaces appearing in (8) as infinite-dimensional Lie groups. The Lie algebra of $\mathrm{Diff}(X)$ is given by smooth vector fields. Indeed, differentiating a family of diffeomorphisms yields a vector field as *tangent vector*. The bracket on this Lie algebra is the bracket of vector fields (which is also called Lie bracket),

$$\mathrm{Lie}(\mathrm{Diff}(X)) = (\Gamma(TX),[\cdot,\cdot]).$$

The discussion surrounding (11) shows that a vector field Y is tangent to $\mathrm{Symp}(X,\omega)\subset\mathrm{Diff}(X)$ if and only if the form $\iota(Y)\omega$ is closed and tangent to $\mathrm{Ham}(X,\omega)$ if and only if it is exact. In other words, we find $\mathrm{Lie}(\mathrm{Symp}(X,\omega))\cong\Omega^1_{\mathrm{cl}}(X)$ and $\cong\Omega^1_{\mathrm{ex}}(X)$ in the Hamiltonian case. But the set of exact one-forms on X is precisely the set of functions on the manifold up to adding a constant to the function. Later we will define the Poisson bracket $\{\cdot,\cdot\}$ on smooth functions, which is exactly the Lie bracket on the Lie algabra of $\mathrm{Ham}(X,\omega)$,

$$\mathrm{Lie}(\mathrm{Ham}(X,\omega) = (C^\infty_0(X),\{\cdot,\cdot\}).$$

Here $C^\infty_0(X)$ denotes smooth functions up to adding a constant.

2.3. **The cotangent bundle.** The cotangent bundle $\pi_Q\colon T^*Q \to Q$ of any smooth manifold Q comes equipped with a *canonical symplectic form* ω_{can}. The so-obtained symplectic manifold

$$(T^*Q, \omega_{\mathrm{can}})$$

is one of the quintessential examples in symplectic geometry. For us it will be crucial, because:

(1) Classical mechanical systems on a smooth manifold Q correspond to a Hamiltonian system on T^*Q. The cotangent bundle is then called *phase space*.
(2) The cotangent bundle (equipped with ω_{can} or a magnetic symplectic form) will serve as a symplectic model space for Lagrangian submersions, which we will study later on.

The symplectic form ω_{can} is exact, meaning that there is a one-form λ_{can} on T^*Q such that $\omega_{\mathrm{can}} = -d\lambda_{\mathrm{can}}$. There are multiple ways to define λ_{can} and ω_{can}. Ours is based on what we call cotangent lifts.

Definition 2.15. *Let Q, Q' be smooth manifolds and $\varphi\colon Q \to Q'$ a diffeomorphism. The* cotangent lift *of φ is defined by*

$$\varphi_!\colon T^*Q \to T^*Q', \quad \eta \mapsto (\varphi^{-1})^*\eta.$$

By this, we mean that for any $Y \in TQ'$, we have

$$\langle (\varphi^{-1})^*\eta, Y\rangle_{Q'} = \langle \eta, (D\varphi^{-1})(Y)\rangle_Q,$$

for all $\eta \in T^*Q$. Here $\langle\cdot,\cdot\rangle$ denotes the pairing of covectors with vectors. We will often drop $\langle\cdot,\cdot\rangle$ from the notation and just write $\eta(Z) = \langle \eta, Z\rangle$. Here are some immediate properties of cotangent lifts:

(1) The cotangent lift preserves fibres, i.e.
$$\pi_{Q'} \circ \varphi_! = \varphi \circ \pi_Q, \tag{12}$$
where $\pi_Q, \pi_{Q'}$ denote the bundle projections of T^*Q and T^*Q', respectively;
(2) it acts linearly on fibres;
(3) it satisfies
$$(\mathrm{id}_Q)_! = \mathrm{id}_{T^*Q}, \quad (\varphi \circ \varphi')_! = \varphi_! \circ \varphi'_!.$$

Now let (U, φ) be a chart $\varphi\colon U \to \mathbb{R}^n$ on a subset $U \subset Q$ of the base space. Denote its image by $V = \varphi(U)$. We can view $T^*V = V \times \mathbb{R}^n \subset \mathbb{R}^{2n}$ by trivializing the bundle T^*V by the sections $dq_i\colon V \to T^*V$, where we call q_i the standard coordinates on $\mathbb{R}^n$. In other words, given $q_* \in V$, we identify

$$p_1 dq_1|_{q_*} + \ldots + p_n dq_n|_{q_*} \in T^*_{q_*}V \quad \text{with} \quad (q_*, p_1, \ldots, p_n) \in V \times \mathbb{R}^n \subset \mathbb{R}^{2n}. \tag{13}$$

Definition 2.16. *Now let (U,φ) be a chart of Q as above. The* cotangent chart *associated to (U,φ) is defined as $\varphi_!\colon T^*U \to T^*V$ followed by the above trivialization $T^*V = V\times\mathbb{R}^n$. We denote the resulting map again by*

$$\varphi_!\colon T^*U \to V\times\mathbb{R}^n, \quad \eta\mapsto(q_1,\dots,q_n,p_1,\dots,p_n).$$

We are now in a position to define the canonical (sometimes called tautological) one-form on T^*Q. Let (U,φ) be a chart and $\varphi_!$ the corresponding cotangent chart. Then we can set

$$(\lambda_{\mathrm{can}})_{(U,\varphi)} = \varphi_!^*(p_1dq_1+\ldots+p_ndq_n)\in\Omega^1(T^*U). \tag{14}$$

Remark 2.17. Here are two points the reader may be confused about:

(1) One-forms are smooth sections of the cotangent bundle,

$$\Omega^1(Q)=\Gamma(T^*Q)=\{\alpha\colon Q\to T^*Q\,|\,\pi_Q\circ\alpha=\mathrm{id}_Q\}.$$

However λ_{can} is not a one-form on Q, but on T^*Q! Therefore it is $\lambda_{\mathrm{can}}\colon T^*Q\to T^*T^*Q$ as a map. In fact, this is part of the reason why it is *canonical* or *tautological*: At $\eta\in T^*Q$ it is given by η itself, which can be canonically viewed in $T^*_\eta T^*Q$, see also Proposition 2.27.

(2) We have slightly abused notation, since by q_i we denote coordinates both on V and on T^*V, compare e.g. (13) and (14). Therefore, we obtain $\pi_Q^*dq_i=dq_i$.

Although the definition (14) is local in Q, it is independent of the choice of (U,φ) and thus yields a globally defined form!

Proposition 2.18. *Let (U,φ) and (U',φ') be two charts of Q. Then*

$$(\lambda_{\mathrm{can}})_{(U,\varphi)}=(\lambda_{\mathrm{can}})_{(U',\varphi')}\quad \text{on } T^*(U\cap U').$$

Proof. First, we prove the following statement. Let $\chi\colon V\to V'$ be diffeomorphism of open sets $V,V'\subset\mathbb{R}^n$. Then the cotangent lift $\chi_!\colon V\times\mathbb{R}^n\to V'\times\mathbb{R}^n$ satisfies

$$\chi_!^*(p_1'dq_1'+\ldots+p_n'dq_n')=p_1dq_1+\ldots+p_ndq_n, \tag{15}$$

where (q_i,p_i) and (q_i',p_i') denote the coordinates determined by (13) on T^*V and T^*V', respectively. By Definition 2.15 of the cotangent lift, we can write

$$\chi_!(q,p)=(\chi(q),(D\chi^{-1})^Tp).$$

We deduce

$$\chi_!^*dq_i'=d(\chi_i(q_1,\dots,q_n))=\textstyle\sum_{j=1}^n\frac{\partial\chi_i}{\partial q_j}dq_j,$$

and

$$p_i'\circ\chi_!=\textstyle\sum_{k=1}^n p_k\frac{\partial(\chi^{-1})_k}{dq_i'}$$

This allows us to compute

$$\begin{aligned}\chi_!^* \textstyle\sum_{i=1}^n p_i' dq_i' &= \textstyle\sum_{i=1}^n \left(\sum_{k=1}^n p_k \frac{\partial(\chi^{-1})_k}{dq_i'}\right)\left(\sum_{j=1}^n \frac{\partial \chi_i}{\partial q_j} dq_j\right)\\ &= \textstyle\sum_{i,j,k=1}^n p_k \frac{\partial(\chi^{-1})_k}{dq_i'} \frac{\partial \chi_i}{\partial q_j} dq_j\\ &= \textstyle\sum_{j,k=1}^n p_k \frac{\partial(\chi^{-1}\circ\chi)_k}{dq_j} dq_j\\ &= \textstyle\sum_{j=1}^n p_j dq_j.\end{aligned}$$

Now to prove the claim of the proposition, we set $\chi = \varphi' \circ \varphi^{-1}$ to deduce $(\varphi_!')^*(\sum_{i=1}^n p_i' dq_i') = \varphi_!^*(\sum_{i=1}^n p_i dq_i)$ from (15). This proves the claim. □

Definition 2.19. *The* canonical one-form $\lambda_{\text{can}} \in \Omega^1(T^*Q)$ *on any cotangent bundle* T^*Q *is defined by*

$$\lambda_{\text{can}}|_U = (\lambda_{\text{can}})_{(U,\varphi)} = \varphi_!^*(p_1 dq_1 + \ldots + p_n dq_n),$$

for any chart (U, φ) *on* Q. *The* canonical symplectic form $\omega_{\text{can}} \in \Omega^2(T^*Q)$ *is defined as* $\omega_{\text{can}} = -d\lambda_{\text{can}}$. *In a cotangent chart, it can be seen as*

$$\omega_{\text{can}}|_U = \varphi_!^*(dq_1 \wedge dp_1 + \ldots + dq_n \wedge dp_n). \tag{16}$$

This form is indeed symplectic. From (16), we can read off that it is closed and non-degenerate. Note that the cotangent charts yield symplectomorphisms of portions $T^*U \subset T^*Q$ of the cotangent bundle with sets of the form $V \times \mathbb{R}^n$ in $\mathbb{R}^{2n}$ equipped with the standard symplectic form $\omega_0 = \sum_i dx_i \wedge dy_i$.

Proposition 2.20. *Let* $\psi_! \colon T^*Q \to T^*Q'$ *be the cotangent lift of any diffeomorphism* $\psi \colon Q \to Q'$. *The canonical one forms are preserved:*

$$\psi_!^* \lambda_{\text{can}}' = \lambda_{\text{can}}.$$

Proof. Let (U, φ) be a chart on Q and (U', φ') be a chart on Q'. Then the map $\chi = \varphi' \circ \psi \circ \varphi^{-1}$ is a diffeomorphism between subsets of $\mathbb{R}^n$. As in the proof of Proposition 2.18, we find $\chi_!^*(\sum_i p_i' dq_i) = \sum_i p_i dq_i$. This proves the claim. □

In classical mechanics, this is extremely helpful. It tells us that we can change coordinate systems on the configuration space Q and still write down the Hamiltonian equation in canonical coordinates.

Another obvious consequence of Proposition 2.20 is that cotangent lifts are symplectomorphisms. Symplectomorphisms preserving a primitive one form (here it is λ_{can}) are sometimes called *exact symplectomorphisms*. We obtain a canonical inclusion

$$\mathrm{Diff}(Q) \hookrightarrow \mathrm{Symp}(T^*Q, \omega_{\text{can}}), \quad \varphi \mapsto \varphi_!. \tag{17}$$

Exercise 2.21 (*). Let $\varphi \in \mathrm{Diff}(Q)$ be a diffeomorphism which is isotopic to the identity through a path in $\mathrm{Diff}(Q)$.

(1) Show that $\varphi_! \in \mathrm{Ham}(T^*Q, \omega_{\mathrm{can}})$. *Hint:* Pick an explicit path $t \mapsto \varphi_t \in \mathrm{Diff}(Q)$ with $\varphi_0 = \mathrm{id}$ and $\varphi_1 = \varphi$ and show that its cotangent lift $(\varphi_t)_!$ is a Hamiltonian flow (of a time-dependent Hamiltonian vector field).
(2) Express the Hamiltonian function generating $t \mapsto (\varphi_t)_!$ in terms of the family of vector fields $t \mapsto Y_t \in \Gamma(TQ)$ generating φ_t.

The symplectomorphisms of $(T^*Q, \omega_{\mathrm{can}})$ coming from cotangent lifts are induced by transformations of the base. Let us now consider translations in the fibre.

Definition 2.22. *Let $\alpha \in \Omega^1(Q)$ be a one-form. The* translation in the fibre along α *is defined by*

$$t_\alpha \colon T^*Q \to T^*Q, \quad \eta \mapsto \eta - \alpha|_{\pi_Q(\eta)}.$$

(The minus here is explained because we want to have a plus in (18)*.)*

Proposition 2.23. *Let $\alpha \in \Omega^1(Q)$ be a one-form. Then $t_\alpha^*\lambda_{\mathrm{can}} = \lambda_{\mathrm{can}} - \pi_Q^*\alpha$ and thus*

$$t_\alpha^*\omega_{\mathrm{can}} = \omega_{\mathrm{can}} + \pi_Q^* d\alpha. \tag{18}$$

In particular, t_α is a symplectomorphism if and only if $\alpha \in \Omega^1_{\mathrm{cl}}(Q)$.

Proof. Let (U, φ) be a chart of Q. We can write $\alpha = \varphi^*(\sum_i \alpha_i dq_i)$. Let us check that the map $\widehat{t}_\alpha = \varphi_! \circ t_\alpha \circ \varphi_!^{-1} \colon V \times \mathbb{R}^n \to V \times \mathbb{R}^n$ is given by

$$\widehat{t}_\alpha(q_*, p) = (q_*, p_1 - \alpha_1, \ldots, p_n - \alpha_n). \tag{19}$$

Indeed, given $\eta \in T^*Q$, we can write $\varphi_!(\eta) = (q_*, p_1, \ldots, p_n) = p_1 dq_1|_{q_*} + \ldots + p_n dq_n|_{q_*}$ as in (13). Therefore $\widehat{t}_\alpha(q_*, p)$ can be written as $(\varphi_! \circ t_\alpha)(\eta) = \varphi_!(\eta - \alpha|_{\pi_Q(\eta)})$. This map preserves the base point and acts linearly on fibres, therefore we can compute $\varphi_!(\alpha|_{\pi_Q(\eta)})$ separately. By definition of the coefficients α_i above, we find that $(\varphi^{-1})^*\alpha = \sum_i \alpha_i dq_i$. Using Definition 2.15, we thus find

$$\varphi_!(\eta - \alpha|_{\pi_Q(\eta)}) = \sum_i p_i dq_i|_{q_*} - \sum_i \alpha_i dq_i|_{q_*} = (q_*, p_1 - \alpha_1, \ldots, p_n - \alpha_n).$$

The identity (19) allows us to compute

$$(\widehat{t}_\alpha)^*(\textstyle\sum_i p_i dq_i) = \sum_i (p_i - \alpha_i) dq_i = \sum_i p_i dq_i - \pi_Q^*(\sum_i \alpha_i dq_i).$$

In the last equation, we have used that

(1) the functions α_i depend only on the base coordinates q_i;
(2) we have made use of the slight abuse of notation in the sense that $dq_i = \pi_Q^* dq_i$, where $dq_i \in \Omega^1(T^*Q)$ on the LHS and $dq_i \in \Omega^1(Q)$ on the RHS. See Remark 2.17.

Using the definition of $\widehat{t_\alpha}$, we find

$$\begin{aligned} t_\alpha^* \lambda_{\mathrm{can}} &= t_\alpha^* \varphi_!^* (\textstyle\sum_i p_i dq_i) \\ &= \varphi_!^* (\textstyle\sum_i p_i dq_i) - \varphi_!^* \pi_Q^* (\textstyle\sum_i \alpha_i dq_i) \\ &= \lambda_{\mathrm{can}} - \pi_Q^* \varphi^* (\textstyle\sum_i \alpha_i dq_i) \\ &= \lambda_{\mathrm{can}} - \pi_Q^* \alpha, \end{aligned}$$

as desired. In the third equality, we have used (12). □

Exercise 2.24 (*). For every $\alpha \in \Omega^1(Q)$, define a unique vector field X_α on T^*Q by

$$\pi_Q^* \alpha = \iota(X_\alpha)\omega_{\mathrm{can}}. \tag{20}$$

Show that its time-one flow $\phi_1^{X_\alpha}$ is exactly t_α. We will crucially use this idea in a later lecture to study the topology of Lagrangian submersions.

Exercise 2.25. When is t_α a Hamiltonian diffeomorphism?

Remark 2.26. Sometimes the *cotangent charts* from Definition 2.16 are dropped from the notation altogether and one just writes

$$\lambda_{\mathrm{can}}|_U = \textstyle\sum_i p_i dq_i, \quad \omega_{\mathrm{can}}|_U = \textstyle\sum_i dq_i \wedge dp_i,$$

This makes perfect sense when we interpret the q_i, p_i as functions of the type $T^*U \to \mathbb{R}$ (obtained from composing the cotangent chart with projection to a coordinate), instead of as coordinates in the image of the cotangent charts. The dq_i are then one forms on T^*U. Physicists call the coordinates q_i, p_i *canonical coordinates.*

On the other hand, there are intrinsic ways of characterizing the canonical one form.

Proposition 2.27. *The canonical one-form λ_{can} can be equivalently defined as follows:*

(1)
$$\lambda_{\mathrm{can}} \colon TT^*Q \to \mathbb{R}, \quad \lambda_{\mathrm{can}}(\xi) = (\pi_Q^*\eta)(\xi) = \eta((\pi_Q)_*\xi), \tag{21}$$
where $\eta \in T^*Q$ denotes the footpoint of the vector $\xi \in T_\eta T^*Q$ and $(\pi_Q)_* \colon TT^*Q \to TQ$ the differential of the bundle projection $\pi_Q \colon T^*Q \to Q$.

(2) λ_{can} is the unique form such that
$$\alpha^* \lambda_{\mathrm{can}} = \alpha, \quad \textit{for all } \alpha \in \Omega^1(Q). \tag{22}$$
Note that this equation makes sense, since $\alpha \colon Q \to T^*Q$ and thus under its pull-back, λ_{can} is a form on Q.

Proof. Both claims follow from local computations in a cotangent chart. We start with *(1)*. Let (U, φ) be a chart of Q and denote by (q_i, p_i) the coordinates in the image of the cotangent chart $\varphi_!$. Recall the identification (13) and write

$$\varphi_!(\eta) = (q, p) = \textstyle\sum_i p_i(\eta) dq_i,$$

and let $\xi' \in T_{(q,p)}T^*V$ This allows us to compute

$$\begin{aligned}\eta((\pi_Q)_*\xi) &= (\varphi_!^{-1}(\textstyle\sum_i p_i(\eta)dq_i))((\pi_Q)_*(\varphi_!^{-1})_*(\varphi_!)_*\xi)\\ &= (\varphi_!^{-1}(\textstyle\sum_i p_i(\eta)dq_i))(\varphi_*^{-1}(\pi_Q)_*(\varphi_!)_*(\xi)\\ &= \textstyle\sum_i p_i(\eta)dq_i((\pi_Q)_*(\varphi_!)_*\xi)\\ &= \textstyle\sum_i p_i(\eta)\pi_Q^* dq_i((\varphi_!)_*\xi)\\ &= (\textstyle\sum_i p_i dq_i)((\varphi_!)_*\xi)\\ &= \varphi_!^*(\textstyle\sum_i p_i dq_i)(\xi) = \lambda_{\mathrm{can}}(\xi),\end{aligned}$$

where in the second equality we have used $\varphi \circ \pi_Q = \pi_Q \circ \varphi_!$, where the third equality follows from the definition of $\varphi_!(\eta) = \eta \circ \varphi_*^{-1}$, and where in the fifth equality follows from the abuse of notation $\pi_Q^* dq_i = dq_i$.

Let $\alpha \in \Omega^1(Q)$. We again work in a chart $(U, \varphi\colon U \to V)$ and the associated canonical coordinates. Let $\alpha_i \in C^\infty(V)$ such that $\alpha = \varphi^*(\sum_i \alpha_i dq_i)$. Under the identification (13), set

$$\widehat{\alpha}\colon V \to T^*V = V \times \mathbb{R}^n, \quad \widehat{\alpha}(q) = (q, \alpha_1(q), \dots, \alpha_n(q)) = \textstyle\sum_i \alpha_i(q) dq_i$$

Then $\varphi_! \circ \alpha = \widehat{\alpha} \circ \varphi$. Now let $Y \in T_qU$ and compute

$$\begin{aligned}(\alpha^*\lambda_{\mathrm{can}})(Y) &= \varphi_!^*(\textstyle\sum_i p_i dq_i)(\alpha_* Y)\\ &= (\textstyle\sum_i p_i dq_i)((\varphi_! \circ \alpha)_* Y)\\ &= (\textstyle\sum_i p_i dq_i)((\widehat{\alpha} \circ \varphi)_* Y)\\ &= \textstyle\sum_i \alpha_i dq_i((\pi_Q)_* \widehat{\alpha}_* \varphi_* Y)\\ &= \varphi^*(\textstyle\sum_i \alpha_i dq_i)(Y) = \alpha(Y).\end{aligned}$$

To show that this property uniquely characterizes λ_{can}, one can use Exercise 2.28, proving that on an open dense subset of TT^*Q we can write any ξ as $\alpha_* Y$. This implies that $\lambda_{\mathrm{can}}(\xi) = \alpha(Y)$ and thus, by continuity, λ_{can} is uniquely determined by this property.

Here is an easier proof of *(2)*: View the map $\alpha\colon Q \to T^*Q$ as $\alpha = t_{-\alpha} \circ 0_Q$, where $0_Q\colon Q \to T^*Q$ is the zero-section. Then we can use Proposition 2.23 to compute

$$\alpha^*\lambda_{\mathrm{can}} = 0_Q^* t_{-\alpha}^* \lambda_{\mathrm{can}} = 0_Q^*(\lambda_{\mathrm{can}} + \pi_Q^*\alpha) = \alpha.$$

Here we have used that λ_{can} vanishes on the zero-section. □

Exercise 2.28. Let $\eta \in T_q^*Q$. Prove that the set

$$\{\alpha_* Y \in T_\eta T^*Q \,|\, Y \in T_qQ,\ \alpha \in \Omega^1(Q),\ \alpha|_q = \eta\}$$

is open and dense in $T_\eta T^*Q$. More precisely, it consists of the complement of the vertical distribution $\ker(\pi_Q)_*$ union the zero vector.

2.4. **Mechanics on manifolds.** Symplectic geometry in cotangent bundles allows us to significantly generalize Hamilton's formalism. Indeed, we can now consider a particle or physical state whose position is constrained to a manifold Q, called *configuration space*.

Definition 2.29. *A* mechanical datum (Q, g, P) *is a triple consisting of a smooth manifold* Q*, a Riemannian metric* g *on* Q*, and a smooth potential function* $P \in C^\infty(Q)$*.*

Classical mechanics on Q is related to symplectic geometry of $(T^*Q, \omega_{\mathrm{can}})$ – this space is called *phase space*. The base coordinate q describes the position of a particle and the fiber component its momentum. Note that although the position is constrained to Q, momenta can take all possible values in the fibre tangent to Q.

Definition 2.30. *Let* (Q, g, P) *be a mechanical datum. The associated* mechanical system *is given by the Hamiltonian system* $(T^*Q, \omega_{\mathrm{can}}, H)$*, where we set*

$$H = H_{(Q,g,P)} \colon T^*Q \to \mathbb{R}, \quad \eta \mapsto \frac{1}{2} g(\eta, \eta) + P(\pi_Q(\eta)). \tag{23}$$

Time evolution of the mechanical system is given by the Hamiltonian flow ϕ^H of $H = H_{(Q,g,P)}$. The first term in (23) is again called *kinetic energy* and the second *potential energy*. Note that we have plugged covectors into the Riemannian metric. We have implicitly used the isomorphism $TQ \to T^*Q$ induced by $Y \mapsto g(Y, \cdot)$.

Example 2.31. For $Q = \mathbb{R}^n$ and $g = g_{\mathrm{Eucl}}$ the standard Euclidean metric on $\mathbb{R}^n$, the mechanical system we obtain coincides with the system defined in 2.1 by Hamilton's equation (4) with the same potential function V. This explains the minus sign in $\omega_{\mathrm{can}} = -d\lambda_{\mathrm{can}}$, which is necessary to recover Hamilton's equations with the traditional sign convention.

For any (Q, g, P) mechanical datum and chart $(U, \varphi \colon U \to V)$ on Q, we can transport the mechanical datum to obtain a new one

$$(V, g_\varphi = (\varphi^{-1})^* g, P_\varphi = (\varphi^{-1})^* P = P \circ \varphi^{-1}) \tag{24}$$

on the subset V of $\mathbb{R}^n$. Here the pull-back metric is defined by $(\psi^* g)(Y, Z) = g(\psi_* Y, \psi_* Z)$ for any two vectors $Y, Z \in TQ$.

Proposition 2.32. *For any mechanical datum* (Q, g, P) *and chart* $(U, \varphi \colon U \to V)$ *on* Q*, let* $(V, g_\varphi, P_\varphi)$ *be its push-forward in the chart as defined in* (24)*. Then their respective Hamiltonian systems are* conjugate*. More precisely:*

$$\varphi_! \circ \phi_t^{H_{(U,g,P)}} = \phi_t^{H_{(V,g_\varphi,P_\varphi)}} \circ \varphi_!. \tag{25}$$

Exercise 2.33 (*). Prove the proposition.

In other words, ever mechanical system locally (in Q) reduces to a mechanical system in $\mathbb{R}^n$ and thus a system which obeys Hamilton's equations (3) by which we have started this course! Furthermore, once we have a solution $t \mapsto q(t) \subset V$ in the image of the chart, we can consider

$$\tau(t) = \varphi^{-1}(q(t)) \in Q. \tag{26}$$

to solve the system on Q.

Actually, there is a much more general fact which goes in the same direction:

Proposition 2.34. *Every Hamiltonian system (X, ω, H) on a $2n$-dimensional symplectic manifold X is locally conjugate to Hamilton's equations in $\mathbb{R}^{2n}$.*

This proof of the proposition uses the classical *Darboux theorem*.

Theorem 2.35 (Darboux theorem)**.** *Let x be a point in a symplectic $2n$-manifold (X, ω). Then there is a neighbourhood U of x and a symplectomorphism $\psi\colon (U, \omega|_U) \to \psi(U) \subset (\mathbb{R}^{2n}, \omega_0)$, where $\omega_0 = \sum_{i=1}^n dx_i \wedge dy_i$.*

Exercise 2.36. Use 2.35 to prove 2.34.

At first glance, it may seem that this result makes canonical coordinates and Proposition 2.32 obsolete. Indeed, any Hamiltonian system can locally be written as Hamilton's equations. Note however that Proposition 2.34 is not compatible with the fibration structure of the cotangent bundle. As a consequence, one cannot just project a solution $t \mapsto (q(t), p(t)) \in \mathbb{R}^{2n}$ to the q-coordinate and then transport it to Q as in (26). Consequently, the system obtained by using Darboux's theorem is not mechanical and completely local, whereas Proposition 2.32 is semi-local: it is local in the base coordinates on Q, but extends to the full fibres over it.

2.5. **Example: Planar pendulum.** Let us now describe the so-called *mathematical* (or *planar*) *pendulum*. Physically speaking, this is an object attached to the end of a rigid arm which moves in the plane $\mathbb{R}^2 = \{(x, y)\}$ equipped with the Euclidean metric $g_{\mathrm{Eucl}} = dx^2 + dy^2$ and which is subjected to a constant gravitational force pointing downwards $F(x, y) = -\partial_y$. Let us fix the mass of the object and the length of the arm to be $= 1$. The potential of the force is $V(x, y) = y$, since we have indeed: $F = -\nabla V$. Note that the Euclidean metric g_{Eucl} is used in the gradient.

The configuration space is not all of $\mathbb{R}^2$, but a circle of unit radius around the point in which the rigid arm is fixed – let's assume the latter is the origin $(0, 0) \in \mathbb{R}^2$. We have

$$Q = \{(x, y) \,|\, x^2 + y^2 = 1\}.$$

The mechanical system (Q, g, V) is obtained by restricting g_{Eucl} and V to the submanifold Q. Let us now work in a chart of Q. To that end, set

$$\chi\colon \mathbb{R} \to Q, \quad \theta \mapsto (\sin\theta, -\cos\theta). \tag{27}$$

It will become clear later why we have chosen this strange parametrization. The map χ is a covering which yields a diffeomorphism $S^1 = \mathbb{R}/(2\pi\mathbb{Z}) \cong Q$. Locally, one can invert it to get charts (U, φ) on Q. Here we will rather work with the globally defined inverse χ of the chart φ then the chart itself. Let us now compute the push-forward datum as in (24) to use

Proposition 2.32. For every local inverse φ of χ, we obtain

$$\begin{aligned} g_\varphi &= (\varphi^{-1})^* g_{\mathrm{Eucl}} \\ &= \chi^* g_{\mathrm{Eucl}} \\ &= d(\sin\theta)^2 + d(-\cos\theta)^2 \\ &= (\cos\theta)^2 d\theta^2 + (\sin\theta)^2 d\theta^2 \\ &= d\theta^2. \end{aligned}$$

The induced potential is

$$V_\varphi(\theta) = (V \circ \chi)(\theta) = -\cos\theta.$$

By Proposition 2.32, we can thus solve the Hamiltonian system associated with the mechanical datum

$$(S^1 = \mathbb{R}/(2\pi\mathbb{Z}), g_{S^1} = d\theta^2, V_{S^1} = -\cos\theta). \tag{28}$$

Its phase space is the cotangent bundle T^*S^1, i.e. the cylinder $\{(\theta, p_\theta) \in S^1 \times \mathbb{R}\}$ equipped with the symplectic form $d\theta \wedge dp_\theta$. The Hamiltonian is

$$H = H_{(S^1, g_{S^1}, V_{S^1})}(\theta, p_\theta) = \frac{p_\theta^2}{2} - \cos\theta. \tag{29}$$

Indeed, the metric g_{S^1} identifies ∂_θ with p_θ and thus $g_{S^1}(p,p) = p_\theta^2$. Recall that autonomous Hamiltonian flows preserve the symplectic form and the Hamiltonian generating them. Therefore, the transformation of T^*S^1 we are looking for preserves the standard area form on the cylinder and the level sets of (29). Let us thus proceed by increasing values of the energy H. For $H < -1$, the level set is empty. The value $H = -1$ is a critical value, and the level set is the point $(\theta, p_\theta) = (0,0)$. This is why we have chosen χ the way we did – the stable equilibrium point of the pendulum is then at the origin. The corresponding orbit is the fixed point where the pendulum is at rest. In the range $H \in (-1,1)$, the values are regular and their level sets are connected closed embedded curves in the cylinder. Thus the system has periodic trajectories which oscillate: At $\theta = 0$ they have momentum $|p_\theta| = \sqrt{2(H + \cos\theta)}$ meaning that all their energy is stored in the kinetic term. They reach their maximal height $H = -\cos\theta$ when $p_\theta = 0$ at which point the energy is purely stored in the potential term. The oscillations can thus be viewed as a continuous trading off of potential energy for kinetic energy (during the fall of the pendulum) and vice-versa (during its ascent). Note that the level sets in the range $H \in (-1,1)$ consist of one single trajectory. This changes at $H = 1$ which is a singular value. Its level is not a manifold, but rather an *eye* whose ends intersect in the cylinder. This level set consists of three pieces of trajectory. The simplest one is the equilibrium point $\theta = \pi \mod 2\pi$ and $p_\theta = 0$. This corresponds to the pendulum standing vertically up. It is unstable and cannot be observed in nature. If $H = 1$ and $p_\theta \neq 0$ there are two possible trajectories distinguished by the sign of p_θ. They are mirror to one another by reflecting the circle with respect to its vertical axis. These trajectories are the only ones which are not periodic. Instead they converge to the point $\theta = \pi$ for time going to $\pm\infty$

but they never reach it. These trajectories cannot be observed in nature either because of friction and material imperfections. The values $H > 1$ are regular and their level sets are disconnected. They consist of two closed embedded curves which are non-contractible in the cylinder, one in the range $p_\theta > 0$ and one in the range $p_\theta < 0$. Physically, these are the trajectories which have enough energy to traverse the north pole $\theta = \pi \mod 2\pi$.

3. Symmetries of Hamiltonian systems

3.1. **Poisson brackets.** To motivate the definition of the Poisson-bracket, recall that we have the following chain of inclusions of groups,

$$\mathrm{Ham}(X,\omega) \subset \mathrm{Symp}(X,\omega) \subset \mathrm{Diff}(X).$$

Let us interpret these as infinite-dimensional Lie groups, without worrying too much about what an infinite dimensional manifold is. By differentiating a smooth curve $t \mapsto \psi_t \in \mathrm{Diff}(X)$ satisfying $\psi_0 = \mathrm{id}$, we obtain a smooth vector field on X. Therefore the Lie algebra of $\mathrm{Diff}(X)$ is given by the space of all vector fields. Its Lie bracket is the Lie bracket for vector fields,

$$\mathrm{Lie}(\mathrm{Diff}(X)) = (\Gamma(TX), [\cdot,\cdot]).$$

Since $\mathrm{Ham}(X,\omega)$ is a subgroup, its Lie algebra is a subalgebra of vector fields. In fact, it is the subalgebra of *Hamiltonian vector fields*

$$\Gamma_{\mathrm{Ham}}(TX) = \{X_H \in \Gamma(TX) \mid H \in C^\infty(X)\},$$

equipped with the Lie bracket for vector fields[1]. The set of Hamiltonian vector fields is in one-to-one correspondence with the set of smooth functions up to adding a constant function $C_0^\infty(X) = C^\infty(X)/\mathbb{R}$. The induced bracket on functions is the *Poisson bracket.* Here is a very direct definition of it. We will come back the interpretation as a Lie bracket in Proposition 3.5.

Definition 3.1. *Let (X,ω) be a symplectic manifold. The* Poisson bracket *is defined by*

$$\{-,-\}\colon C^\infty(X) \times C^\infty(X) \to C^\infty(X), \quad \{F,G\} = \omega(X_F, X_G).$$

We say that $F, G \in C^\infty(X)$ Poisson-commute *if $\{F,G\} = 0$.*

By Hamilton's equation, the Poisson bracket can be rewritten as

$$\{F,G\} = dF(X_G) = -dG(X_F), \tag{30}$$

meaning that it measures the rate of change of one of the functions under the Hamiltonian flow of the other. Remarkably, this rate of change is anti-symmetric under exchanging the functions.

Remark 3.2. The Poisson bracket only depends on the functions up to adding constants and is thus well-defined as a map $C_0^\infty(X) \times C_0^\infty(X) \to C_0^\infty(X)$. Indeed, the Hamiltonian vector field X_F depends only on dF, and not on F itself.

[1] In particular, the bracket of two Hamiltonian vector fields is again Hamiltonian!

Proposition 3.3. *The Poisson bracket satisfies the following three properties*

(1) $\{F, G\} = -\{G, F\}$ (anti-symmetry),
(2) $\{F, GH\} = H\{F, G\} + G\{F, H\}$ (Leibniz rule),
(3) $\{F, \{G, H\}\} + \{G, \{H, F\}\} + \{H, \{F, G\}\} = 0$ (Jacobi identity),

for all $F, G, H \in C^\infty(X)$.

Exercise 3.4. Prove these properties. *Hint:* For the latter property, compute first

$$\{F, \{G, H\}\} = -\mathcal{L}_{X_F}(\omega(X_G, X_H))$$

and then use Cartan's magic formula, $\mathcal{L}_X = d\iota(X) + \iota(X)d$.

These three properties can be used as axioms to define abstract Poisson brackets. An abstract Poisson bracket does not necessarily come from a symplectic structure, but instead it induces a decomposition of X into a set of symplectic leaves. The study of the geometry one obtains in this way is called *Poisson Geometry*. We will not pursue this any further here and our Poisson bracket always comes from an underlying symplectic structure.

Proposition 3.5. *The Poisson bracket has the following basic properties for all* $F, G \in$

(1) If $F, G \in C^\infty(X)$ *Poisson-commute, then the functions are constant along each other's Hamiltonian flows,*

$$F \circ \phi_t^G = F, \quad G \circ \phi_t^F = G;$$

(2)

$$X_{\{F,G\}} = -[X_F, X_G]$$

In particular, the map $F \mapsto X_F$ *yields an isomorphism of Lie algebras,*

$$(C^\infty(X), \{\cdot, \cdot\}) \to (\Gamma_{\mathrm{Ham}}(TX), [\cdot, \cdot]);$$

(3) If $F, G \in C^\infty(X)$ *Poisson-commute, then their Hamiltonian flows commute, meaning that for all times* $t, s \in \mathbb{R}$, *we have*

$$\phi_t^F \circ \phi_s^G = \phi_s^G \circ \phi_t^H.$$

Proof. To prove *(1)*, apply $\frac{d}{dt}$ to find $dF(X_G)$. The claim follows from (30).

The proof of *(2)* relies on the Jacobi identity form Proposition 3.3. Recall that any vector field is characterized by how it acts on smooth functions as a derivation. In particular, the Lie bracket can be defined by

$$[X_F, X_G]H = X_F(X_G H) - X_G(X_F H), \quad \text{for all } H \in C^\infty(X).$$

Again, we use that $X_G H = dH(X_G) = \{H, G\}$ to rewrite this expression as $\{\{H, G\}, F\} - \{\{H, F\}, G\} = \{F, \{G, H\}\} + \{G, \{H, F\}\}$. By the Jacobi identity, this is equal to $-\{H, \{F, G\}\} = -X_{\{F,G\}}H$. To summarize, we obtain $[X_F, X_G]H = -X_{\{F,G\}}H$ for all smooth functions H and this proves the claim.

For the proof of *(3)*, recall that if $[Y, Z] = 0$ for any pair of vector fields Y, Z, then their flows $t \mapsto \phi_t^Y$ and $s \mapsto \phi_s^Z$ commute, $\phi_t^Y \circ \phi_s^Z = \phi_s^Z \circ \phi_t^Y$. Let us stress that this is a general fact about *smooth* (and not necessarily Hamiltonian) flows. Now if $\{F, G\} = 0$, we can use *(2)* to find that $[X_F, X_G] = 0$. Since the Hamiltonian flows of F, G are the flows of the vector fields X_F, X_G meaning, in our notation, that $\phi_t^{X_F} = \phi_t^F$, the claim follows. □

Exercise 3.6. Find $F, G \in C^\infty(X)$ whose Hamiltonian flows commute, but which do not Poisson-commute, themselves. In other words the converse to point *(3)* in Proposition 3.5 does not hold.

Let us now discuss how the Poisson bracket is useful in *solving* a Hamiltonian system (X, ω, H) with symmetry. This discussion is *crucial* for later topics in this lecture. Recall that finding an explicit solution is impossible most of the time. Using energy conservation $H \circ \phi_t^H = H$ allows us to look for solutions in the level sets $H^{-1}(h_0)$ of H. This removes one degree of freedom and bring us to dimension $2n - 1$. Although this helps, it is hardly satisfying, except in the case $n = 1$ (as for example in the case of the planar pendulum). Now assume that there is an auxiliary G with $\{H, G\} = 0$.

Then we can use *(1)* in Proposition 3.5 to find that G preserved under ϕ_t^H, too! In other words, we can now restrict our attention to joint level sets of the form

$$H^{-1}(h_0) \cap G^{-1}(g_0) = \{H = h_0, G = g_0\}.$$

Whenever these are transverse, i.e. when dH, dG are linearly independent (it is obviously of no use to consider $G = 2H$ for example), we lose two degrees of freedom.

We can use *(3)* in Proposition 3.5 to get a finer idea of what solutions look like in the joint level sets. Indeed, if we have one solution $t \mapsto \phi_t^H(x_0)$, we find a one-parametric family of solutions by transporting it by the Hamiltonian flow of G. Indeed, transporting the initial condition x_0 by ϕ_s^G and then solving the system $t \mapsto \phi_t^H(\phi_s^G(x_0))$ is the same as solving it for x_0 and then applying $\phi_s^G(x_0)$.

Remark 3.7. The function G is sometimes called *(first) integral of* (X, ω, H), in the sense that it helps us *integrate* the system, i.e. solve it.

Noether's theorem can be roughly stated as follows: *For every symmetry of a system, there is an associated preserved quantity.* Using the Poisson bracket formalism, this is almost a tautology. Indeed, the *system* is the Hamiltonian flow ϕ_t^H, the *symmetry* is the Hamiltonian flow ϕ_s^G. The *conserved quantity* is simply the Hamiltonian G itself!

Remark 3.8. A Hamiltonian system (X, ω, H) can have at most $n - 1$ independent symmetries. If it has this maximal amount of symmetry i.e. if there are smooth functions $G_2, \ldots, G_n$ such that

$$\{H, G_i\} = \{G_i, G_j\} = 0$$

for all i, j and such that $dH, dG_2, \ldots, dG_n$ are linearly independent (at least on an open dense subset of X), then the system is called *(completely) integrable.* In that case the candidate space for solutions has dimension $2n - n = n$. The structure of these candidate spaces is elucidated by the so-called *Arnol'd–Liouville theorem.*

3.2. **Hamiltonian circle actions.** As we have seen, it is extremely useful to find a first integral G of a Hamiltonian system (X, ω, H) because it allows us to restrict our attention to the level set $\{G = g_0\} \subset X$, thus reducing the dimension of the space under consideration by one. Furthermore, the solutions of the system appear in one-dimensional families on that level set. It is thus natural to ask whether one can quotient out by the transformation inducing these one-dimensional families.

Under the additional assumption that the Hamiltonian flow of G generates a circle action, this idea is spectacularly useful. Indeed, in that case the quotient $\{G = g_0\}/S^1$ is itself a symplectic manifold, provided the action on the level set $\{G = g_0\}$ is free. Furthermore, this *symplectic quotient* carries a *residual Hamiltonian system* whose dynamics is intimately related to the dynamics of the original system (X, ω, H). Note however that the quotient $\{G = g_0\}/S^1$ has dimension $2n - 2$, meaning that we have actually reduced the number of degrees of freedom by two! Before moving on to discussing this *symplectic reduction* procedure, let us have a look at Hamiltonian circle actions.

Definition 3.9. *A smooth group action $\psi \colon S^1 \times X \to X$ on a symplectic manifold (X, ω) is called* Hamiltonian group action *if there is an (autonomous) Hamiltonian G such that $\psi_\theta = \psi(\theta, -)$ is equal to ϕ_t^G for all $t \in \mathbb{R}$ and $\theta = [t] \in S^1 = \mathbb{R}/\mathbb{Z}$. The Hamiltonian G is then called* moment map *of the action.*

In other words, any autonomous Hamiltonian G whose flow satisfies $\phi_1^G = \mathrm{id}$ induces a Hamiltonian S^1-action on X. We switch between viewing the action as a map $\psi \colon S^1 \times X \to X$ and a map $\theta \mapsto \psi_\theta \in \mathrm{Ham}(X, \omega)$.

Here are some examples of Hamiltonian S^1-actions.

Example 3.10. The system $(\mathbb{R}^2, \omega_0, G(x, y) = \pi(x^2 + y^2))$ defines a Hamiltonian S^1-action. It is just the rotation of the plane and thus has one fixed point at the origin $(x, y) = (0, 0)$. Note that this is a special case of the harmonic oscillator.

Exercise 3.11. Let $(\mathbb{R}^{2n}, \omega_0 = \sum_i dx_i \wedge dy_i)$ be equipped with the Hamiltonian

$$G_\alpha(x_1, y_1, \ldots, x_n, y_n) = \pi\alpha_1(x_1^2 + y_1^2) + \ldots + \pi\alpha_n(x_n^2 + y_n^2)$$

for $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$. For which vectors $\alpha \in \mathbb{R}^n$ does this define a Hamiltonian S^1-action? For which vectors α can G_α be rescaled by some $c \in \mathbb{R}$ to cG_α defining a Hamiltonian S^1-action?

Example 3.12. Let $S^1 \times \mathbb{R} = \mathbb{R}^2/(x, y) \sim (x + 1, y)$ be equipped with the symplectic form $dx \wedge dy$. This space is symplectomorphic to $(T^*S^1, \omega_{\mathrm{can}})$. Then the moment map $G(x, y) = y$ induces a Hamiltonian S^1-action by translation in the x-direction. Note that it does not have any fixed points.

Example 3.13. Let $S^2 = \{x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3$ be equipped with the scaling ω_{S^2} of the natural area form which satisfies $\int_{S^2} \omega_{S^2} = 2$. Then the height function

$$G \colon S^2 \to \mathbb{R}, \quad G(x, y, z) = z$$

defines a Hamiltonian S^1-action fixing the North and South poles $(0,0,\pm 1) \in S^2$.

Exercise 3.14 (*). Prove the claims in the previous example. *Hint:* Define a parametrization

$$\chi\colon S^1 \times (-1,1) \to S^2 \setminus \{(0,0,\pm 1)\},$$

and consider the pull-back system $(S^1 \times (-1,1), \chi^*\omega_{S^2}, G \circ \chi)$.

Another major source of examples are lifts of smooth actions to cotangent bundles.

Proposition 3.15. *Let $\theta \mapsto \psi_\theta$ be a* smooth *circle action on a smooth manifold Q. Then its cotangent lift $\theta \mapsto (\psi_\theta)_!$ defines a Hamiltonian circle action on the cotangent bundle $(T^*Q, \omega_{\mathrm{can}})$. We denote this action by $\psi_!\colon S^1 \times T^*Q \to T^*Q$.*

Proof. Exercise 2.21 shows that the lift $\theta \mapsto (\psi_!)_\theta = (\psi_\theta)_!$ is a Hamiltonian flow. Since $(\psi_!)_1 = (\psi_1)_! = (\mathrm{id}_Q)_! = \mathrm{id}_{T^*Q}$, we find that this yields a Hamiltonian S^1-action. □

Exercise 3.16. Among Examples 3.10, 3.12 and 3.13, which of them are a lift of a smooth action to a cotangent bundle?

Example 3.17. Let $\mathbb{R}^2 = \{(q_1, q_2)\}$ be equipped with the smooth S^1-action given by the standard rotation,

$$\psi_\theta(q_1, q_2) = (q_1 \cos 2\pi\theta - q_2 \sin 2\pi\theta, q_2 \cos 2\pi\theta + q_1 \sin 2\pi\theta).$$

Then its cotangent lift $\psi_!$ acts by simultaneous rotation on the coordinate pairs $q = (q_1, q_2)$ and $p = (p_1, p_2)$ of the cotangent bundle $T^*\mathbb{R}^2 = \{(q_1, q_2, p_1, p_2)\}$,

$$(\psi_!)_\theta(q,p) = (\psi_\theta(q), \psi_\theta(p)). \tag{31}$$

One way of seeing this is by noting that, since we are in $\mathbb{R}^2$, we can write the cotangent lift in coordinates

$$(\psi_\theta)_!(q,p) = (\psi_\theta(q), (D\psi_\theta^{-1})^T p).$$

But $\psi_\theta\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear map! Therefore, we find $D\psi_\theta = \psi_\theta$. Furthermore, since it is a rotation, we have $\psi_\theta^{-1} = \psi_\theta^T$, proving (31). Another approach is computing the Hamiltonian vector field. One can use Exercise 2.21 to find that the moment map generating the lifted action $(\psi_!)_\theta$ is

$$G(q,p) = p(2\pi(q_1\partial_{q_2} - q_2\partial_{q_1})) = p_1 q_2 - p_2 q_1,$$

since the vector field $Y = 2\pi(q_1\partial_{q_2} - q_2\partial_{q_1})$ generates the rotation on the base $\mathbb{R}^2 = \{(q_1, q_2)\}$. See also Example 2.5. The preserved quantity G of this rotation has a physical interpretation as *angular momentum.*

3.3. **Interlude: Geodesics.** Let (Q, g) be a Riemannian manifold and ∇ its Levi-Civita connection. A smooth curve $\gamma\colon [t_0, t_1] \to Q$ is called *geodesic* if it satisfies

$$\nabla_{\dot\gamma}\dot\gamma = 0. \tag{32}$$

Such curves are locally length-minimizing. For every tangent vector $Y \in TQ$, there is a unique geodesic γ with $\dot\gamma(0) = Y$ which exists at least locally. Existence of such geodesics for all t and Y is called *geodesic completeness*. The Hopf-Rinow theorem tells us that geodesic completeness is equivalent to *metric completeness* of the space Q equipped with the metric induced by g.

Definition 3.18. *Let (Q, g) be a geodesically complete Riemannian manifold. The* geodesic flow $\Gamma\colon \mathbb{R} \times TQ \to TQ$ *is defined by*

$$\Gamma(t, Y) = \Gamma_t(Y) = \dot{\gamma}(t),$$

where $\gamma\colon \mathbb{R} \to Q$ is the unique geodesic with $\dot{\gamma}(0) = Y$.

In the framework of classical mechanics (32) can be interpreted as *the acceleration of γ being zero*. In light of Newton's theorem, this means there is no external force acting on the particle, i.e. that $P = 0$. This is also sometimes called *free particle*. In fact, one can make this rigorous.

Exercise 3.19 (!). Show that the geodesic flow is Hamiltonian in the following sense: Let $g^{\#}\colon TQ \to T^*Q$ denote the isomorphism defined by $g^{\#}(Y) = g(Y, \cdot)$. Then the geodesic flow is conjugate to the Hamiltonian flow,

$$\phi_t^H = g^{\#} \circ G_t \circ (g^{\#})^{-1},$$

where $H = H_{(Q,g,P=0)}$ denotes the mechanical Hamiltonian with vanishing potential, $P = 0$.

For the sake of this course, we can take this to be the definition of the geodesic flow.

Definition 3.20. *Let (Q, g) be a Riemannian manifold. We call $\phi_t^{H_{(Q,g,P=0)}}$ the* geodesic flow*. The actual* geodesics *are obtained by projecting to the base space, $\gamma(t) = \pi_Q(\phi_t^H(x_0))$.*

Some people call this the *co-geodesic flow* but we will not bother doing this distinction. The Hamiltonian point of view on geodesics is extremely helpful to deal with symmetries of (Q, g). This is illustrated by the following.

Exercise 3.21 (*). The goal of this exercise is to prove *Clairaut's relation* for the geodesics on surfaces of revolution. For any smooth function $z \mapsto f(z) \in \mathbb{R}_{>0}$, we consider its *surface of revolution* around the z-axis in $\mathbb{R}^3$,

$$\Sigma_f = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = f(z)^2\} \subset \mathbb{R}^3.$$

Convince yourself that the name of this surface is well-deserved. We equip Σ_f with the Riemannian metric g_f induced by the ambient Euclidean metric $g_{\mathrm{Eucl}} = dx^2 + dy^2 + dz^2$. Prove that if γ is a geodesic, then the quantity

$$f(\gamma_z(t)) \cos \alpha(t)$$

is constant in t. Here $\gamma_z(t)$ denotes the z-component of the curve γ and $\alpha(t)$ the angle between $\dot{\gamma}(t)$ and the circle of constant height $\{z = \gamma_z(t)\} \subset \Sigma_f$. Does the converse hold?

3.4. **Symplectic reduction.** Note that Hamiltonian circle actions preserve the level sets of the moment map generating them. Therefore, we can consider the quotient space of any level set on which S^1 acts *freely*, i.e. for any $x \in G^{-1}(g_0)$, if $\psi_\theta(x) = x$ then $\theta \in S^1$ is the neutral element. Equivalently, the stabilizer of every point in $G^{-1}(g_0)$ is trivial. The quotient turns out to be a symplectic manifold in its own right. This procedure is called *symplectic reduction.*

Theorem 3.22. *Let (X, ω) be equipped a moment map $G \in C^\infty(X)$ generating a Hamiltonian S^1-action which acts freely on the level set $Z = G^{-1}(g_0)$. Then its quotient $X_{g_0} = G^{-1}(g_0)/S^1$ carries a unique symplectic form ω_{g_0} satisfying*

$$p^*\omega_{g_0} = \omega|_{TZ}, \tag{33}$$

where $p\colon Z \to X_{g_0}$ denotes the natural quotient map.

Proof. Let us first prove that X_{g_0} is a smooth manifold. Firstly, Z is a smooth manifold, since g_0 is a regular value. Indeed, suppose by contradiction that it contains a critical point $x_0 \in G^{-1}(g_0)$. Then $X_G|_{x_0} = 0$ and x_0 is fixed under the Hamiltonian flow of G. This is in contradiction with the action being free. The S^1-action on the level set is free by hypothesis and automatically proper, since S^1 is compact. Therefore the quotient space is a smooth manifold.

Let us now show that the quotient carries a symplectic form satisfying (33). For all $x \in X_{g_0}$ and $Y, Z \in T_x X_{g_0}$, we can set

$$(\omega_{g_0})_x(Y_1, Y_2) = \omega_{\widehat{x}}(\widehat{Y}_1, \widehat{Y}_2), \tag{34}$$

where $\widehat{x} \in X$ is a lift of x and $\widehat{Y}_1, \widehat{Y}_2$ are lifts of Y_1, Y_2 in the sense that $p(\widehat{x}) = x$ and $p_*(\widehat{Y}_i) = Y_i$. Such lifts exist since X_{g_0} is by definition the quotient of Z and thus p is a surjective submersion. If we can prove that ω_{g_0} as defined by (34) is well-defined and symplectic, then we are done. Indeed, the form (34) satisfies (33).

Let $(\widehat{x}', \widehat{Y}_1', \widehat{Y}_2')$ be another triple lifting (x, Y_1, Y_2) in the above sense. Let us show that the RHS of (34) yields the same for that triple. First note that there is $\theta_0 \in S^1$ such that $\psi_{\theta_0}(\widehat{x}) = \widehat{x}'$, but ψ_θ acts by Hamiltonian diffeomorphisms and therefore preserves the symplectic form ω,

$$\omega_{\widehat{x}}(\widehat{Y}_1, \widehat{Y}_2) = (\psi_{\theta_0}^*\omega)_{\widehat{x}}(\widehat{Y}_1, \widehat{Y}_2) = \omega_{\widehat{x}'}((\psi_{\theta_0})_*\widehat{Y}_1, (\psi_{\theta_0})_*\widehat{Y}_2).$$

We thus need to compare $(\psi_{\theta_0})_*\widehat{Y}_1$ to $\widehat{Y}_1'$. To that end, note that

$$p_*((\psi_{\theta_0})_*\widehat{Y}_1 - \widehat{Y}_1') = p_*(\widehat{Y}_1) - Y_1 = 0,$$

since $p \circ \psi_\theta = p$. This proves that $(\psi_{\theta_0})_*\widehat{Y}_1 - \widehat{Y}_1'$ is tangent to the fibre direction and thus there is λ such that

$$\lambda X_G = (\psi_{\theta_0})_*\widehat{Y}_1 - \widehat{Y}_1'.$$

It follows that

$$\omega((\psi_{\theta_0})_*\widehat{Y}_1, \cdot) = \omega(\widehat{Y}_1' + \lambda X_G, \cdot) = \omega(\widehat{Y}_1', \cdot) + \lambda dG(\cdot),$$

where the latter term vanishes on $TZ = \ker dG$. Hence we obtain a well-defined differential two-form $\omega_{g_0} \in \Omega^2(X_{g_0})$ satisfying (33). This form is closed since $p^* d\omega_{g_0} = (d\omega)|_{TZ} = 0$ by (33) and p is a submersion. Non-degeneracy requires some more thought. Recall that non-degeneracy means that $\omega^\#\colon TX \to T^*X$ defined by $\omega^\#(Y) = \iota(Y)\omega$ is an isomorphism. This is the case if and only if

$$\ker \omega^\# = \{Y \in TX \mid \iota(Y)\omega = 0\}$$

is trivial. Although ω is non-degenerate on X, its restriction $\omega|_{TZ}$ *is* degenerate! Indeed,

$$(\iota(X_G)\omega)_{TZ} = (dG)_{TZ} = 0,$$

since, again, $TZ = \ker dG$. For dimensional reasons (think about this!), we find that X_G is the only degenerate direction, $\ker \omega^\# = \operatorname{span} X_G$. In other words, symplectic reduction precisely mods out by the only degeneracy of $\omega|_{TZ}$ and thus the form descends to a non-degenerate one on the quotient.

To formally prove this, we take $Y \in TX_{g_0}$ such that $\omega_{g_0}^\#(Y) = \iota(Y)\omega_{g_0} = 0$ and show that $Y = 0$. Picking $\widehat{Y}$ a lift of Y, we can lift the form,

$$0 = p^*(\iota(Y)\omega_{g_0}) = \omega_{g_0}(Y, p_*(\cdot)) = \omega_{g_0}(p_*\widehat{Y}, p_*(\cdot)) = (\iota(\widehat{Y})\omega)|_{TZ}.$$

Meaning that $\widehat{Y}$ is in the kernel of $\omega|_{TZ}$, and thus a multiple of X_G. Therefore $p_*\widehat{Y} = Y = 0$. □

Remark 3.23. The projection $p\colon Z \to X_{g_0}$ defines a fibre bundle which is given the structure of an S^1-principal bundle by the action generated by G.

Definition 3.24. *We call the so-obtained symplectic manifold* (X_{g_0}, ω_{g_0}) *the* reduced space (at the level $G = g_0$). *It is also sometimes called* symplectic quotient *or* Marsden–Weinstein quotient. *If the hypotheses of Theorem 3.22 are satisfied, we say that* X admits (symplectic) reduction at the level $G = g_0$.

Example 3.25. Let $(\mathbb{R}^{2n+2}, \omega_0)$ be equipped with the Hamiltonian

$$G = \pi(x_0^2 + y_0^2 + \ldots + x_n^2 + y_n^2).$$

Setting $z_j = x_j + iy_j$, we can identify $\mathbb{R}^{2n+2} = \mathbb{C}^{n+1}$. In this notation G generates the Hamiltonian S^1-action $\psi_\theta = \phi_\theta^G$ given by

$$\psi_\theta(z_0, \ldots, z_n) = (e^{-2\pi i\theta} z_0, \ldots, e^{-2\pi i\theta} z_n).$$

This is sometimes called the *diagonal circle action*. The level set for $g_0 > 0$ is given by the sphere of radius $\sqrt{g_0/\pi}$,

$$G^{-1}(g_0) = S^{2n+1}\left(\sqrt{g_0/\pi}\right).$$

The action is free on the spheres, since, for every point $z = (z_0, \dots, z_n)$ on it, there is at least one $z_j \neq 0$ and ψ_θ acts by rotation on z_j. Therefore, we can perform symplectic reduction. The quotient $X_{g_0} = S^{2n+1}/S^1$ is diffeomorphic to complex projective space $\mathbb{C}P^n$ and the induced symplectic form is called the *Fubini–Study form* $\omega_{\mathrm{FS}} \in \Omega^2(\mathbb{C}P^n)$. Note that there is a freedom in choosing $c > 0$. This freedom corresponds to rescaling the Fubini–Study form, and different authors use different conventions.

Let us now return to the setting of a Hamiltonian system (X, ω, H) which has a first integral G. Assume furthermore that G is the moment map of a Hamiltonian S^1-action. Whenever G admits symplectic reduction, the reduced space (X_{g_0}, ω_{g_0}) itself carries a Hamiltonian system H_{g_0} induced by H which recovers some of the dynamics of the original system.

Proposition 3.26. *Let (X, ω, H) be a Hamiltonian system and G be the moment map of a Hamiltonian S^1-action on X such that $\{H, G\} = 0$. Assume furthermore that (X, ω) admits symplectic reduction at the level $G = g_0$. Then the Hamiltonian on the reduced space*

$$H_{g_0} \colon X_{g_0} \to \mathbb{R}, \qquad \text{defined by } H_{g_0} \circ p = H, \tag{35}$$

has the following property

$$p \circ \phi_t^H = \phi_t^{H_{g_0}} \circ p, \tag{36}$$

where $p \colon G^{-1}(g_0) \to X_{g_0}$ denotes the natural quotient map.

Exercise 3.27 (*). Prove that (35) determines a well-defined Hamiltonian H_{g_0} and prove the proposition.

Definition 3.28. *In the above setup, we call $(X_{g_0}, \omega_{g_0}, H_{g_0})$ the* residual system *of (X, ω, H) at the level $G = g_0$.*

Remark 3.29. All of this holds in the much more general framework of *Hamiltonian group actions*. These are a certain type of group action by a (compact connected) Lie group G_0 on a symplectic manifold generated by a moment map

$$\mu \colon (X, \omega) \to \mathfrak{g}^*,$$

taking values in the dual to the Lie algebra $\mathfrak{g}^*$. This map is G_0-equivariant with respect to the co-adjoint action of G_0 on $\mathfrak{g}^*$. Symplectic reduction can still be carried out to yield a symplectic manifold $\mu^{-1}(g_0)/G_0$ whenever $g_0 \in \mathfrak{g}^*$ is invariant under the co-adjoint action and the action on the level set is free.

3.5. **Example: Spherical pendulum.** The spherical pendulum is a physical system consisting of a rigid arm with one end fixed at a point (the origin, say) of $\mathbb{R}^3$ and with a weight attached at the other end. It is subjected to a homogeneous gravitational force field. We assume that the arm has no mass and unit length, that there is no friction, and that the weight is concentrated at one point and has unit mass.

In our language, this yields a mechanical system with the following mechanical datum: $Q = S^2 = \{(q_1, q_2, q_3) \,|\, q_1^2 + q_2^2 + q_3^2 = 1\}$ equipped with the round metric g_{S^2}, i.e. with the restriction of $g_{\mathrm{Eucl}} = dq_1^2 + dq_2^2 + dq_3^2$ to the tangent space of S^2, and with the potential $P(q_1, q_2, q_3) = q_3$. Recall that the Hamiltonian is given by $H(\eta) = \frac{1}{2} g_{S^2}(V_\eta, V_\eta) + P(\pi_{S^2}(\eta))$, where we have used the metric to define $V_\eta \in TS^2$ as the unique vector such that $\eta = g_{S^2}(V_\eta, \cdot)$. Under the identification

$$T^*S^2 \cong TS^2 = \{(q,p) = (q_1, q_2, q_3, p_1, p_2, p_3) \in \mathbb{R}^3 \times \mathbb{R}^3 \,|\, \|q\| = 1, p \cdot q = 0\}, \tag{37}$$

we can write

$$H(q,p) = \frac{p_1^2 + p_2^2 + p_3^2}{2} + q_3.$$

Remark 3.30. Writing it out in these coordinates is dangerous – in fact many authors avoid it altogether. The solution to the system is *not* just obtained by looking at Hamilton's equations with respect to the (q_i, p_i), since this would yield a solution in $(T^*\mathbb{R}^3, \omega_{\mathrm{can}})$, not in the subspace defined by (37).

The spherical pendulum has *rotational symmetry*, i.e. we can rotate S^2 around the q_3-axis without changing the system. Let us prove this formally. Define the smooth S^1-action

$$\begin{aligned} \psi \colon S^1 \times S^2 &\to S^2, \\ (\theta, (q_1, q_2, q_3)) &\mapsto (q_1 \cos 2\pi\theta - q_2 \sin 2\pi\theta, q_1 \sin 2\pi\theta + q_2 \cos 2\pi\theta, q_3) \end{aligned} \tag{38}$$

The cotangent lift $\psi_!$ of this action yields a Hamiltonian circle action on T^*S^2, see Proposition 3.15. Since ψ

(1) acts by isometries on (S^2, g_{S^2}), and
(2) preserves the potential P,

we find that its lift preserves H, i.e. $H \circ (\psi_!)_\theta = H$.

Exercise 3.31. Formalize this argument.

Definition 3.32. *We call the Hamiltonian $G \in C^\infty(T^*S^2)$ generating the Hamiltonian circle action $\theta \mapsto (\psi_\theta)_!$* angular momentum.

Since $dH(X_G) = 0$ and we find that

$$\{H, G\} = 0.$$

Thus $(T^*S^2, \omega_{\mathrm{can}}, H)$ has a first integral G.

Remark 3.33. Since we are in dimension four, this is the maximal amount of symmetry one can have, and thus the spherical pendulum is an *integrable system.* Furthermore, this integral generates a Hamiltonian circle action, meaning that we are in a very favorable situation!

The Hamiltonian S^1-action $\psi_!$ is generated by the function

$$G(q,p) = 2\pi(q_1 p_2 - q_2 p_1).$$

This is angular momentum with respect the rotation around the q_3-axis.

Exercise 3.34. Prove that G generates $\psi_!$. *Hint:* After you're done, look at your so-called *proof* and find the mistake. Now fix it.

Let us now analyze the dynamics of the pendulum. A fixed point of ϕ_t^H is called *equilibrium* and the equilibria are equal to the critical points of H. Indeed, by Hamilton's equation, these are exactly the points where the vector field X_H vanishes. In the case of a mechanical system (Q, g, P),

$$\mathrm{Crit}\, H = \{(q_0, 0) \in T^*Q \mid q_0 \in \mathrm{Crit}\, P \subset Q\},$$

i.e. every equilibrium point lies on the zero-section and corresponds to a critical point of the potential P.

Exercise 3.35. Convince yourself of this.

In the case of the spherical pendulum, the potential $P = q_3$ has the north pole $N = (0, 0, 1)$ and the south pole $S = (0, 0, -1)$ as critical points (Recall that we need consider $P|_{S^2}$ as a function on S^2, not on $\mathbb{R}^3$). Therefore, the equilibria are $(N, 0), (S, 0) \subset T^*S^2$. These points correspond to the pendulum being at rest and in a vertical position pointing upwards and downwards respectively.

Since G generates a Hamiltonian S^1-action, we can perform symplectic reduction as in Theorem 3.22 to obtain a residual Hamiltonian system as in Proposition 3.26. Recall that the system admits reduction on the level $G^{-1}(g_0)$ if the S^1-action on that level is *free*. A necessary condition is that g_0 be a regular value. Critical points of G are fixed points of the action $\psi_!$ and those are exactly $(S, 0)$ and $(N, 0)$. The fact that $\mathrm{Crit}\, H = \mathrm{Crit}\, G$ is a coincidence! The only *critical value* of G is therefore $g_0 = 0$. It is therefore extremely useful to decompose T^*S^2 into invariant subsets

$$T^*S^2 = \{G = 0\} \sqcup \{G \neq 0\},$$

which we analyze separately. We start with $G = 0$.

Proposition 3.36. *A point $(q_0, p_0) \in T^*S^2$ has vanishing angular momentum, $G(q_0, p_0) = 0$ if and only if its orbit $\gamma(t) = \pi_{S^2}(\phi_t^H(q_0, p_0)) \in S^2$ is contained in a great circle through the North and South Poles, $N = (0, 0, 1)$ and $S = (0, 0, -1)$. Furthermore, the spherical pendulum restricts to a planar pendulum on every such circle.*

Proof. The *if* direction follows either from direct computation or by interpreting G in terms of the vector product,

$$G(q, p) = (p \times q) \cdot e_3. \tag{39}$$

Recall that $p \times q$ is orthogonal to the plane spanned by $p, q \in \mathbb{R}^3$ and thus if p, q are contained in a plane containing N, S then $p \times q$ is orthogonal to e_3 and $G = 0$. The *only if* direction is slightly more involved. Let $(q_0, p_0) \in T^*S^2$ be a point with $G(q_0, p_0) = 0$ and let $\Pi \subset \mathbb{R}^3$ be a plane spanned by q_0, p_0. If $p_0 = 0$, then we pick such a plane containing the poles N, S. If $p_0 \neq 0$, then the plane is uniquely defined as $\Pi = (p_0 \times q_0)^{\perp}$. By (39), we find that the condition $G(q_0, p_0) = 0$ shows that this plane contains the North and South Poles. We need to prove that the full orbit is contained in Π.

Because of rotational symmetry, we can assume without loss of generality that $\Pi = \{q_1 = 0\}$. Denote by $R^\Pi\colon S^2 \to S^2$ the reflection $(q_1, q_2, q_3) \mapsto (-q_1, q_2, q_3)$ through Π. Its cotangent lift $R^\Pi_!\colon T^*S^2 \to T^*S^2$ is given by

$$(q_1, q_2, q_3, p_1, p_2, p_3) \mapsto (-q_1, q_2, q_3, -p_1, p_2, p_3)$$

and preserves H, meaning that $H \circ R^\Pi_! = H$. Note that $R^\Pi_!(q_0, p_0) = (q_0, p_0)$. We claim that the fixed point set

$$\operatorname{Fix} R^\Pi_! = \{(q,p) \in T^*S^2 \mid R^\Pi_!(q,p) = (q,p)\} = \{q_1 = p_1 = 0\}$$

is preserved under the flow ϕ^H_t. Indeed, since $R^\Pi_!$ is a symplectomorphism, the discussion surrounding (7) shows that

$$(R^\Pi_!)_* X_H = X_{H \circ R^\Pi_!} = X_H.$$

This proves that X_H is tangent to $\operatorname{Fix} R^\Pi_!$ and hence its flow preserves the great circle $\pi_{S^2}(\operatorname{Fix} R^\Pi_!) \subset S^2$. Restricting the metric, the potential and the symplectic form to $\operatorname{Fix} R^\Pi_!$ yields the planar pendulum. Indeed, this can be checked by picking an explicit embedding

$$i\colon S^1 \times \mathbb{R} \to T^*S^2, \quad (\theta, p_\theta) \mapsto (\sin\theta, -\cos\theta, p_\theta \cos\theta, p_\theta \sin\theta)$$

which parametrizes $\operatorname{Fix} R^\Pi_!$ and computing the pull-back system $(S^1 \times \mathbb{R}, i^*\omega_{\mathrm{can}}, i^*g_{S^2}, i^*P)$ to find that it agrees with the planar pendulum. □

Exercise 3.37. The spherical pendulum is not just a collection of planar pendula: Which part of the (*only if* part of the) proof fails when $G \neq 0$?

The set $\{G = 0\}$ consists of an S^1-family of planar pendula. These intersect in the equilibria at the North and South poles.

Remark 3.38. The circle action generated by G acts on the orbits of the planar pendula in an interesting way. Recall from §2.5 that the planar pendulum has a critical value at $H = 1$ below which its level sets are circles and above which they are a disjoint union of circles. A half-turn ϕ^G_π maps a circle below the critical value to itself, whereas, above the critical value, it interchanges the the two connected components of the level set!

Let us move to the invariant subset $\{G \neq 0\}$. Trajectories with $G \neq 0$ do not intersect the poles S, N. Indeed, for every $(q_0, p_0) \in T^*_S S^2 \sqcup T^*_S S^2$ we have $G(q_0, p_0) = 0$. Since ψ_θ acts freely on $S^2 \setminus \{S, N\}$, we deduce that the lift acts freely on $G^{-1}(g_0) \subset T^*S^2 \setminus (T^*_S S^2 \sqcup T^*_S S^2)$. Therefore, for every $g_0 \neq 0$, the system admits symplectic reduction by G. In order to identify the so-obtained quotient spaces, it is convenient to introduce a chart. We choose spherical coordinates

$$\begin{aligned} \varphi\colon A = S^1 \times (0, 1/2) &\to S^2 \setminus \{S, N\}, \\ (\theta, \alpha) &\mapsto (\sin 2\pi\alpha \cos 2\pi\theta, \sin 2\pi\alpha \sin 2\pi\theta, \cos 2\pi\alpha), \end{aligned}$$

which yield a diffeomorphism omitting the poles. Note that this diffeomorphism is S^1-equivariant with respect to the rotation in the first factor on the domain and the action ψ_θ on the target. The cotangent lift yields an exact symplectomorphism

$$\varphi_! \colon T^*A \to T^*S^2 \setminus (T^*_N S^2 \cup T^*_S S^2). \tag{40}$$

On $T^*A = T^*(\mathbb{R}/\mathbb{Z} \times (0, 1/2)) = T^*(\mathbb{R}/\mathbb{Z}) \times T^*(0, 1/2)$, we choose the natural coordinates

$$(\theta, \alpha, p_\theta, p_\alpha) \in \mathbb{R}/\mathbb{Z} \times (0, 1/2) \times \mathbb{R} \times \mathbb{R}. \tag{41}$$

Recall that, formally speaking, these are defined by

$$\eta = p_\theta(\eta) d\theta_{(\theta(\eta),\alpha(\eta))} + p_\alpha(\eta) d\alpha_{(\theta(\eta),\alpha(\eta))}.$$

for all $\eta \in T^*A$. In the cotangent chart, the Hamiltonians $H_\varphi = H \circ \varphi_!, G_\varphi = G \circ \varphi_!$ look as follows

$$\begin{aligned} H_\varphi(\theta, \alpha, p_\theta, p_\alpha) &= \frac{1}{8\pi^2}\left(\frac{p_\theta^2}{(\sin 2\pi\alpha)^2} + p_\alpha^2\right) + \cos 2\pi\alpha, \\ G_\varphi(\theta, \alpha, p_\theta, p_\alpha) &= 2\pi p_\theta. \end{aligned}$$

Exercise 3.39 (*). Prove this. *Hint:* For G, you should not compute anything. For H, use the discussion surrounding Proposition 2.32. You'll need some familiarity with Riemannian metrics for that.

Remark 3.40. The Hamiltonian system $(T^*A, \omega_{\text{can}}, H_\varphi)$ is *not complete*!! Recall that this means that the flow $\phi_t^{H_\varphi}$ may not be defined for all times, depending on which point we start with. Indeed, solutions of $(T^*S^2, \omega_{\text{can}}, H)$ intersect at least one of the fibre $T^*_N S^2, T^*_S S^2$ if and only if $G = 0$. Since this is an equivalence, the system becomes complete when we restrict our attention to $\{G_\varphi \neq 0\} = T^*A \setminus \{p_\theta = 0\}$. Since this set is equal to $\varphi_!^{-1}(G \neq 0)$, it is exactly the subset we are interested in, here!

We are now ready to perform symplectic reduction with respect to G.

Proposition 3.41. *The residual Hamiltonian system of $(T^*S^2, \omega_{\text{can}}, H)$ at the level $G = 2\pi g_0 \neq 0$ is given by*

$$\begin{aligned} (T^*S^2)_{g_0} &\cong T^*(0, 1/2) = (0, 1/2) \times \mathbb{R} = \{(\alpha, p_\alpha)\}, \\ (\omega_{\text{can}})_{g_0} &= \omega_{\text{can}}^{T^*(0,1/2)} = d\alpha \wedge dp_\alpha, \\ H_{g_0} &= \frac{p_\alpha^2}{8\pi^2} + \frac{g_0^2}{8\pi^2(\sin 2\pi\alpha)^2} + \cos 2\pi\alpha. \end{aligned}$$

Proof. Since $G \neq 0$, the level set $G = 2\pi g_0 \neq 0$ is completely contained in the image of $\varphi_!$ and thus we can work with the residual system of $(T^*A, \omega_{\text{can}}, H_\varphi)$ at the level $G_\varphi = 2\pi g_0$, instead. In the coordinates (41), the level set is

$$G_\varphi^{-1}(2\pi g_0) = \mathbb{R}/\mathbb{Z} \times (0, 1/2) \times \{g_0\} \times \mathbb{R},$$

and the action on it is the obvious S^1-action on the first factor. Therefore symplectic reduction simply eliminates the first factor in $T^*A = T^*(\mathbb{R}/\mathbb{Z}) \times T^*(0,1/2)$. Since the symplectic form also splits, $\omega_{\mathrm{can}} = d\theta \wedge dp_\theta + d\alpha \wedge dp_\alpha$ we find that the reduced space is $(T^*(0,1/2), d\alpha \wedge dp_\alpha)$. The residual Hamiltonian H_φ is simply the restriction to the level set, which yields the expression we have claimed. Note that its well-definedness on the symplectic quotient is reflected in the fact that H_φ does not depend on θ. $\square$

The Hamiltonian H_{g_0} is again mechanical, and its potential is

$$P_{\mathrm{eff},g_0}(\alpha) = \frac{g_0^2}{8\pi^2(\sin 2\pi\alpha)^2} + \cos 2\pi\alpha. \tag{42}$$

This is called the *effective potential*, because a part of the kinetic energy term of H has been absorbed into the potential.

Definition 3.42. *A point $x \in G^{-1}(g_0)$ is called* relative equilibrium *if it maps to a critical point of the residual Hamiltonian H_{g_0}, i.e. if $p(x) \in \operatorname{Crit} H_{g_0}$.*

A point $x \in G^{-1}(g_0)$ is a relative equilibrium if and only if there is $\lambda \in \mathbb{R}$ such that $X_H|_x = \lambda X_G|_x$. Physically, this means that the system H has the same flow (up to time reparameterization) as the symmetry. In our case this means that the solutions of H are obtained by rotating the sphere. For every $g_0 \neq 0$, the effective potential P_{eff,g_0} has one critical point which is a global minimum. We call the critical point $\alpha_{\mathrm{crit}}(g_0) \in (0,1/2)$. Actually, a more detailed analysis of P_{eff,g_0} shows that

$$\alpha_{\mathrm{crit}}(g_0) \in (1/4, 1/2)$$

with $\alpha_{\mathrm{crit}}(g_0)$ approaching $1/4$ as $g_0 \to \pm\infty$. This means that when we increase angular momentum, the rotational solutions coming from the relative equilibria approach the equator of the sphere. And there are no such solutions in the upper hemisphere!

The residual Hamiltonian $H_{g_0} = \frac{p_\alpha^2}{8\pi^2} + P_{\mathrm{eff},g_0}(\alpha)$ thus has one critical point $(\alpha_{\mathrm{crit}}(g_0), 0)$ which corresponds to a global minimum. Let $m(g_0)$ be the value of that minimum, i.e.

$$m(g_0) = H_{g_0}((\alpha_{\mathrm{crit}}(g_0), 0)) = P_{\mathrm{eff},g_0}(\alpha_{\mathrm{crit}}(g_0)).$$

The graph of the function

$$m \colon \mathbb{R} \setminus \{0\} \to \mathbb{R}, \quad g_0 \mapsto m(g_0)$$

is roughly shaped like a parabola with $\lim_{g_0 \searrow 0} m(g_0) = \lim_{g_0 \nearrow 0} m(g_0) = -1$ and thus we extend m by continuity. The resulting function is *not smooth* at $g_0 = 0$.

Remark 3.43. Let us interpret this physically. For a given angular momentum g_0, the pendulum cannot have energy less than $m(g_0)$. Indeed, on the level set $G^{-1}(g_0)$ the energy H restricts to the function H_{g_0} which satisfies $H_{g_0} \geqslant m(g_0)$. Furthermore, for fixed angular momentum g_0 and energy h_0, the trajectory $\gamma(t) \in S^2$ is trapped in an annulus

$\{(\theta, \alpha) \mid \alpha \in [\alpha_{\min}, \alpha_{\max}]\} \subset S^2$ where the segment $[\alpha_{\min}, \alpha_{\max}]$ is determined by projecting the level set $H_{g_0}^{-1}(h_0)$ to the α-axis.

Now consider the map

$$F = (G, H) \colon T^*S^2 \to \mathbb{R}^2 = \{(g, h)\}. \tag{43}$$

This is what we call an *integrable system* below. Note that the fibres of this map are invariant sets of ϕ_t^H, so they contain the orbits of the Hamiltonian system we are trying to solve! A common theme in the study of integrable systems is trying to understand the geometry of this map, i.e. its critical points and the topology of its fibres.

Proposition 3.44. *The image of F as defined by* (43) *is given by*

$$\operatorname{im} F = \{(g, h) \in \mathbb{R}^2 \mid h \geq m(g)\}.$$

Its critical values are

$$\operatorname{Critval} F = \{(g, h) \in \mathbb{R}^2 \mid h = m(g)\} \cup \{(0, 1)\}.$$

Its fibres of F are as follows

1. *$F^{-1}(0, -1)$ consists of a point which is the equilibrium point $(S, 0)$;*
2. *$F^{-1}(0, 1)$ is a two-sphere with one self-intersection (this is also sometimes called a* pinched torus*). The self-intersection point is the equilibrium point $(N, 0)$ and its complement consists of an S^1-family of non-periodic orbits as in the planar pendulum;*
3. *$F^{-1}(g, m(g))$ for $g \neq 0$ is a circle corresponding to a relative equilibrium type orbit;*
4. *all other fibres are smooth two-tori.*

In particular, all of its fibres are compact and connected.

Proof. For $g = 0$, the system restricts to planar pendula, which have energy $H \in [-1, +\infty)$ in our convention. For $g \neq 0$, note that $H|_{G^{-1}(g)} = H_g \circ p$ under the symplectic quotient map $p \colon G^{-1}(g) \to (T^*S^2)_g$, therefore

$$\operatorname{im} H|_{G^{-1}(g)} = \operatorname{im} H_g = [m(g), +\infty)$$

as claimed.

As for the critical points, note that $x \in \operatorname{Crit} F$ if and only if there are $\lambda_1, \lambda_2 \in \mathbb{R}$ such that

$$\lambda_1 X_F|_x + \lambda_2 X_G|_x = 0$$

This happens either if the rank of $dF|_x$ is zero or one. The case of rank zero happens to coincide with the *equilibrium points* $x \in \{(S, 0), (N, 0)\}$. Their critical values are $(0, -1)$ and $(0, 1)$. In particular, those are contained in the level $g = 0$. There are no other critical points in this level, which is why we can suppose $g \neq 0$ from now on. In that case the rank of $dF|_x$ is at least one since there are no critical points of the H, G as individual functions in that case. Therefore if x is critical we can suppose that

$$X_H|_x = \lambda X_G|_x,$$

meaning precisely that x is a *relative equilibrium*! By the above analysis, this happens only for $p(x) = (\alpha_{\mathrm{crit}}(g), 0)$. Again, since H_g lifts to the restriction $H|_{G^{-1}(g)}$, we find that the critical value is $(g, m(g))$.

To determine the topology of the fibres, let us start with the case $g \neq 0$, where we can perform symplectic reduction. Recall that $p\colon G^{-1}(g) \to (T^*S^2)_g$ is a principal circle bundle. Furthermore, we have

$$F^{-1}(g,h) = G^{-1}(g) \cap H^{-1}(h) = p^{-1}(H_g^{-1}(h)).$$

By the analysis of the function H_g on the reduced space, we know that $H_g^{-1}(m(g))$ is a point, while all other levels above $m(g)$ are diffeomorphic to circles. Therefore the fibre $F^{-1}(g, m(g))$ is a circle and the fibres $F^{-1}(g,h)$ for $h > m(g)$ are tori. Indeed, they are the total space of a principal circle bundle over a circle and thus diffeomorphic to a torus. The case $g = 0$ follows from interpreting the level set $G = 0$ as an S^1-family of planar pendula and is left to the reader. □

Remark 3.45. The fact that the fibres over regular values of F are tori is not a coincidence. We will see below that any compact connected fibre over the regular value of an integrable system is a smooth torus.

Exercise 3.46. Carry out a similar analysis for the geodesic flow on surfaces of revolution (see also Exercise 3.21): When can one do symplectic reduction? Are the residual Hamiltonian systems themselves mechanical? Are the fibres of the system compact?

4. Integrable systems and the Arnol'd–Liouville theorem

4.1. **Basics of integrable systems.** We start by the definition of a commuting/integrable system.

Definition 4.1. *Let (X, ω) be a symplectic $2n$-manifold. We call a smooth map $F = (F_1, \dots, F_n)\colon X \to \mathbb{R}^n$* commuting system *if*

(1) its fibres $F^{-1}(b)$ are connected,
(2) the set of regular values is open and dense in the image of F,
(3) its components Poisson-commute pairwise, i.e. $\{F_i, F_j\} = 0$ for all i, j.

If, additionally, such a system has compact fibres, we call it integrable system. *We denote its image by $B = \operatorname{im} F \subset \mathbb{R}^n$ and the set of regular values by $B_{\mathrm{reg}} \subset B$.*

The full name should probably be *completely integrable Hamiltonian system*, but we will abbreviate here. The reader should think of the coordinate functions F_i as Hamiltonian functions. The Hamiltonian flows of these preserve the fibres $F^{-1}(b)$, see Proposition 3.5. We think of F as the base of a singular fibration, the fibres of which are invariant subsets of the commuting/integrable system. As always, unless we state otherwise, we only consider systems whose flows are *complete*, i.e. defined for all times t.

Exercise 4.2. Show that n is the maximal number of such functions one can have, i.e. prove that if there is a map $F\colon X^{2n} \to \mathbb{R}^k$ having the properties of a commuting/integrable system, then $k \leqslant n$.

Examples 4.3. (1) By what we have proved in §3.5, the spherical pendulum is an integrable system.

(2) On $(\mathbb{R}^{2n} = \mathbb{C}^n, \omega_0)$, the map
$$F(z_1, \dots, z_n) = (\pi|z_1|^2, \dots, \pi|z_n|^2)$$
is an integrable system. Its image is $B = \mathbb{R}^n_{\geqslant 0}$ and its regular values lie in $B_{\mathrm{reg}} = \mathbb{R}^n_{>0}$. The fibres are the so-called *product tori*, $F^{-1}(b_1, \dots, b_n) = \{\pi|z_i|^2 = b_i\}$, which are compact and connected. This is a very special system, because its flows $\phi_t^{F_i}$ (given by rotation in the i-th coordinate plane) are Hamiltonian S^1-actions which fit together to an effective Hamiltonian T^n-action.

(3) Again on $(\mathbb{R}^{2n}, \omega_0 = \sum_i dx_i \wedge dy_i)$ the maps
$$G(x, y) = x, \quad F(x, y) = y$$
(where we have denoted $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $y = (y_1, \dots, y_n) \in \mathbb{R}^n$) both define commuting systems, which are not integrable systems. The fibres are copies of $\mathbb{R}^n$. The Hamiltonian flows are just given by coordinate translation $X_{G_i} = -\partial_{y_i}$ and $X_{F_i} = \partial_{x_i}$.

(4) Let us turn the previous example into an integrable system. To that end, we consider the x_i-variables modulo 1 and call them θ_i. Let $(\mathbb{R}/\mathbb{Z})^n \times \mathbb{R}^n = \{(\theta_i, b_i)\}$ be equipped

with the symplectic form $\omega_0 = \sum_i d\theta_i \wedge db_i$. Let $J(\theta, b) = b$. The coordinate function $J_i(\theta, b) = b_i$ generates the standard rotation in the i-th circle. This space is symplectomorphic to the cotangent bundle T^*T^n of the n-torus $T^n = (\mathbb{R}/\mathbb{Z})^n$ equipped with ω_{can}. The map G is not the projection to the base T^n but to the fibre of T^*T^n. This only works because $T^*T^n \to T^n$ is a trivial bundle. Let $U \subset \mathbb{R}^n$ be an open subset. Then we set

$$\mathcal{T}(U) = J^{-1}(U) = \{(\theta, b) \,|\, b \in U\},$$

equipped with the restriction of ω_0. Since the b-coordinate is preserved under the flows $\phi_t^{J_i}$, we obtain a complete Hamiltonian system

$$J \colon \mathcal{T}(U) \to U \subset \mathbb{R}^n, \tag{44}$$

which will serve as model system.

Remark 4.4. Examples (2) and (4) are very special! Not only are they integrable systems, but, additionally, the flows $\phi_t^{F_i}$ generate Hamiltonian S^1-actions. Since these actions commute, they fit together to a T^n-torus action and this action is effective. In this situation F is the *moment map* of a toric action.

Given a commuting/integrable system, it is easy to define new ones. In fact, it suffices to post-compose the system with *any diffeomorphism* of the base. This fact and the transformation behaviour of the system under diffeomorphisms will be very important throughout the rest of the course.

Proposition 4.5. *Let $F \colon X \to \mathbb{R}^n$ be a commuting/integrable system and $f \colon \mathbb{R}^n \to \mathbb{R}^n$ a diffeomorphism. Then the map $G = f \circ F$*

(1) is again a commuting/integrable system (with the same set of fibres!);
(2) for all $x \in X$, has Hamiltonian flows given by

$$\phi_t^{G_i}(x) = \phi_{t\xi_1}^{F_1} \circ \ldots \circ \phi_{t\xi_n}^{F_n}(x), \qquad \text{for all } t \in \mathbb{R}, \tag{45}$$

where $\xi_j = \frac{\partial f_i}{\partial b_j}(F(x))$, in the standard coordinates $(b_1, \ldots, b_n) \in \mathbb{R}^n$. When viewed as functions of $x \in X$, the ξ_i only depend on the base point $b = F(x)$.

Proof. The set of fibres of F and $G = f \circ F$ agree and thus the fibres of G are connected, too. The set of critical values are related by $\operatorname{Critval} G = f(\operatorname{Critval} F)$, therefore the regular values of G are open and dense in the image $\operatorname{im} G = f(\operatorname{im} F)$. For $x \in X$ with $b = F(x)$, we can compute

$$\begin{aligned} dG_i|_x &= d(f_i \circ F)|_x \\ &= \sum_j \frac{\partial (f_i \circ F)}{\partial b_j}(x) db_j|_{F(x)} \\ &= \sum_{j,k} \frac{\partial f_i}{\partial b_k}(F(x)) \frac{\partial F_k}{\partial b_j}(x) db_j|_b \\ &= \sum_k \frac{\partial f_i}{\partial b_k}(b) dF_k|_x. \end{aligned}$$

Using Hamilton's equation and the bilinearity of ω, we find

$$X_{G_i}|_x = \sum_k \frac{\partial f_i}{\partial b_k}(b) X_{F_k}|_x. \tag{46}$$

Therefore the Poisson brackets can be written as

$$\{G_i, G_j\} = \sum_{k,l} \frac{\partial f_i}{\partial b_k} \frac{\partial f_j}{\partial b_l} \{F_k, F_l\} = 0,$$

since all Poisson brackets $\{F_k, F_l\}$ vanish. There is a subtlety here: We have used that the coefficients $\partial_{b_k} f_i$ depend only on the base coordinate $b = F(x)$ and can thus be treated as elements $\in \mathbb{R}$. If this was not the case, the Leibniz rule would make terms of the form $\{\partial_{b_k} f_i, F_k\} \neq 0$ appear. It follows that G is again an integrable system. Equation (45) follows from integrating (46). □

Remark 4.6. Should we think of $G = f \circ F$ as *new* integrable system? After all, the set of fibres of F and G coincide as we have only *reparametrized* their values in $\mathbb{R}^n$. So topologically, these systems agree. But in terms of dynamics, they do not, as (45) shows.

Remark 4.7. On the fibre $F^{-1}(b) = G^{-1}(f(b)) \subset X$, the vector fields $(X_{G_1}, \dots, X_{G_n})$ are obtained from the vector fields $(X_{F_1}, \dots, X_{F_n})$ by multiplying with

$$Df(b)^T = \left(\frac{\partial f_j}{\partial b_i}(b) \right)_{i,j}.$$

Again, we stress that this transformation only depends on b, not on the specific point in the fibre.

4.2. **The Arnol'd–Liouville Theorem.** We now turn to one of the main results of this course.

Theorem 4.8 (Arnol'd–Liouville)**.** *Let $F\colon X \to B \subset \mathbb{R}^n$ be an integrable system with regular values B_{reg} and $b_0 \in B_{\mathrm{reg}}$. Then there is a neighbourhood $U \subset B_{\mathrm{reg}}$ of b_0 and a* *pair* (ψ, f) *consisting of a diffeomorphism* $f\colon U \to V \subset \mathbb{R}^n$ *and of a symplectomorphism* $\psi\colon F^{-1}(U) \to \mathcal{T}(V)$ *such that*

$$f \circ F = J \circ \psi. \tag{47}$$

Here $J\colon \mathcal{T}(V) \to V$ *denotes the integrable system defined in Example 4.3* (4).

Properly understanding this result requires some unpacking:

(1) It claims first of all that the fibre $F^{-1}(b_0)$ has a neighbourhood which is symplectomorphic to $\mathcal{T}(V) = T^n \times \mathbb{R}^n$ equipped with the standard symplectic form $\sum_i d\theta_i \wedge db_i$.

(2) The symplectomorphism ψ maps fibres of F to fibres of J, in the sense that

$$\psi(F^{-1}(b)) = J^{-1}(f(b)) = T^n \times \{f(b)\} \subset \mathcal{T}(V).$$

In particular, every regular fibre of F is a *torus* (called Arnol'd–Liouville torus) and the restriction $F|_{F^{-1}(U)}\colon F^{-1}(U) \to U$ is a trivial torus bundle.

(3) In terms of dynamics, these tori are invariant subsets. Furthermore, as we will see in Exercise 4.9 below, the flows $\phi_t^{F_i}$ are linear in the natural coordinates on $F^{-1}(b)$ induced by ψ. The Arnol'd–Liouville theorem *does not say* that the flows $\phi_t^{F_i}$ are conjugate to the flows $\phi_t^{J_i}$. Indeed, the latter are always periodic of unit period, whereas the former are not.

Exercise 4.9 (*). In the setup of the Arnol'd–Liouville theorem, use its statement to prove that for all $x \in U$ the Hamiltonian flows of F and J are related by

$$(\psi \circ \phi_t^{F_i})(x) = \phi_{t\xi_1}^{J_1} \circ \ldots \circ \phi_{t\xi_n}^{J_n}(\psi(x))$$

for some ξ_i depending only on $b = F(x) \in B$.

The space model space $\mathcal{T}(V) = (\mathbb{R}/\mathbb{Z})^n \times \mathbb{R}^n$ has natural coordinates $\{(\theta_i, b_i)\}$. The coordinates induced on $F^{-1}(U)$ by ψ via the Arnol'd–Liouville theorem are called *action-angle coordinates.*

Definition 4.10. *The functions*

$$J_i \circ \psi\colon F^{-1}(U) \to \mathbb{R}, \quad x \mapsto b_i(\psi(x))$$

are called action coordinates. *The functions*

$$F^{-1}(U) \to S^1 = \mathbb{R}/\mathbb{Z}, \quad x \mapsto \theta_i(\psi(x))$$

are called angle coordinates. *A regular fibre* $F^{-1}(b) \subset X$ *of* F *is called* Arnol'd–Liouville torus.

4.3. **Interlude: Integrable systems on surfaces.** Let us have a closer look at integrable systems and Hamiltonian circle actions on symplectic surfaces, i.e. in dimension two.

Exercise 4.11 (*). Let (X, ω) be a symplectic surface, i.e. $\dim X = 2$. Furthermore let $F\colon X \to \mathbb{R}$ be a Hamiltonian and $b_0 \in \mathbb{R}$ a regular value such that $F^{-1}(b_0) \subset X$ is compact and connected (i.e. a circle).

(1) Show that there is a neighbourhood $I \subset \mathbb{R}$ of b_0 such that the Hamiltonian flow of F is periodic on $F^{-1}(I)$.
(2) Show that there is a well-defined *period function* $P\colon I \to \mathbb{R} \setminus \{0\}$, satisfying the following: If $\phi^F_\theta(x) = x$ for some $x \in F^{-1}(I)$ then $\theta = kP(F(x))$ for some $k \in \mathbb{Z}$;
(3) Is the period function unique with respect to the latter property?
(4) Show that on such a neighbourhood I, there is an invertible function $f\colon I \to I' \subset \mathbb{R}$ such that $f \circ F\colon F^{-1}(I) \to I'$ is the moment map of a Hamiltonian S^1-action, i.e. that

$$\phi_1^{f\circ F}(x) = x, \quad \text{for all } x \in F^{-1}(I).$$

Hint: For this part of the exercise, you may assume some regularity of P without proving it. We will see later that P is actually C^∞.

Actually, the moment map of a Hamiltonian circle action on a surface has the following beautiful interpretation.

Proposition 4.12. *Let F be a Hamiltonian on a symplectic surface (X, ω) and let $I = (b_0, b_1) \subset \mathbb{R}$ be a subset of the regular values of F such that $F^{-1}(b)$ is diffeomorphic to a circle for all $b \in I$. Then F induces a Hamiltonian S^1-action on $F^{-1}(I)$ if and only if*

$$\int_{F^{-1}(b_0,b)} \omega = b - b_0, \quad \textit{for all } b \in (b_0, b_1). \tag{48}$$

Exercise 4.13 (*). The goal of this exercise is to prove the proposition.
(1) Show that there is a *smooth section* $\gamma\colon (b_0, b_1) \to F^{-1}(b_0, b_1)$ of F, meaning that

$$F(\gamma(s)) = s, \quad \text{for all } s \in (b_0, b_1).$$

(2) Prove that the map

$$h\colon \mathbb{R} \times (b_0, b_1) \to F^{-1}(b_0, b_1), \quad (t, s) \mapsto \phi^F_t(\gamma(s))$$

is a local diffeomorphism and satisfies $h^*\omega = dt \wedge ds$.
(3) Is h a symplectomorphism?
(4) Prove the Proposition. *Hint:* For the *if* direction, interpret the integral in (48) in terms of the period function P from Exercise 4.11.

Exercise 4.14 (*). Using the above, prove the Arnol'd–Liouville theorem in dimension two.

4.4. **The fundamental action and the period lattice.** Recall that if $\{H, G\} = 0$, then any joint level set $(H, G)^{-1}(b_0)$ is preserved by both the flow generated by H and the flow generated by G. If b_0 is a regular value, these flows are linearly independent and thus we expect the level set to decompose into two-dimensional invariant subsets. For integrable/commuting systems, something very remarkable happens: The dimension of the level set agrees with the dimension of its invariant subsets. Therefore, under the connectedness assumption, the level sets consist of one single orbit! Let us now formalize this.

Let $F\colon X \to B$ be an integrable system, and $U \subset B_{\mathrm{reg}}$ an open subset. Define

$$\Phi = \Phi^F \colon \mathbb{R}^n \times F^{-1}(U) \to F^{-1}(U), \quad (\tau_1, \ldots, \tau_n, x) \mapsto \phi^{F_1}_{\tau_1} \circ \ldots \circ \phi^{F_n}_{\tau_n}(x) \tag{49}$$

Since the components F_i Poisson-commute pairwise, we obtain $\Phi^F_{\tau+\tau'} = \Phi^F_\tau \circ \Phi^F_{\tau'}$ for any $\tau, \tau' \in \mathbb{R}^n$. Here we have denoted $\Phi_\tau = \Phi(\tau, -)$. Thus the map (49) defines a smooth $\mathbb{R}^n$-action on $F^{-1}(U)$. Note that this action is *symplectic* in that it satisfies $(\Phi^F_\tau)^*\omega = \omega$.

Proposition 4.15. *The restriction of Φ^F to every regular fibre $F^{-1}(b)$ yields a transitive $\mathbb{R}^n$-action. The stabilizer of this restricted action is discrete.*

Recall that a topological group is called discrete if its unit is an isolated point, i.e. has a neighbourhood which intersects the group only in that point.

Proof. The restriction is well-defined since Φ^F preserves F. To show that the action is transitive, let $x \in F^{-1}(b)$ and let $\mathcal{O}_x$ be its orbit. We need to show that $F^{-1}(b) = \mathcal{O}_x$. The orbit $\mathcal{O}_x \subset F^{-1}(b)$ is open: Indeed, the Hamiltonian vector fields $X_{F_1}, \ldots, X_{F_n}$ are linearly independent and the fibre $F^{-1}(b)$ is n-dimensional. Therefore the map $\tau \mapsto \Phi^F_\tau(x)$ is a local diffeomorphism. Any point $y \in F^{-1}(b)$ in the complement of $\mathcal{O}_x$ belongs to its own orbit $\mathcal{O}_y$, which is also open by the previous argument. Hence the orbits are closed and open in the connected set $F^{-1}(b)$. The stabilizer is discrete again because $\tau \mapsto \Phi^F_\tau(x)$ is a local diffeomorphism and thus there is a neighbourhood of $0 \in \mathbb{R}^n$ which does not contain any other elements in the stabilizer. $\square$

Now let $x \in F^{-1}(b)$ be a point in a regular fibre of an integrable system. We now know that the map

$$\mathbb{R}^n \to F^{-1}(b), \quad \tau \mapsto \Phi_\tau(x).$$

is a local diffeomorphism and that it descends to an honest diffeomorphism $\mathbb{R}^n/\operatorname{stab}(x) \to F^{-1}(b)$. Furthermore, the stabilizer only depends on b, since the fibre consists of a single orbit[2].

To understand the topology of the fibre $F^{-1}(b)$, we are thus left with understanding discrete subgroups of $\mathbb{R}^n$. The following classification of discrete subgroups of $\mathbb{R}^n$ is classical, see e.g. [1, Lemma 3, p.276].

Theorem 4.16. *Let $\Lambda \subset \mathbb{R}^n$ be a discrete subgroup. Then there is a basis $v_1, \ldots, v_n \subset \mathbb{R}^n$ such that Λ is spanned (over $\mathbb{Z}$) by $v_1, \ldots, v_k$.*

Definition 4.17. *A discrete subgroup of $\mathbb{R}^n$ is called a* lattice*. The integer $k \in \mathbb{N}$ as in Theorem 4.16 is called the* rank *of Λ.*

This yields a complete understanding of the topology of fibres of integrable and commuting systems.

[2] In general the stabilizer of different points in the same orbit are related by conjugation. Since $\mathbb{R}^n$ is abelian, the stabilizer is actually constant

Proposition 4.18. *Every regular fibre of an integrable system is diffeomorphic to the n-torus T^n.*

Proof. As we have discussed above, it follows from Proposition 4.15 that $F^{-1}(b) \cong \mathbb{R}^n/\Lambda_b$ for some lattice $\Lambda_b \subset \mathbb{R}^n$. By Theorem 4.16, we find that $F^{-1}(b)$ is diffeomorphic to $T^k \times \mathbb{R}^{n-k}$, where k is the rank of Λ_b. Recall that the fibres of integrable systems are compact, meaning that $k = n$ and $F^{-1}(b) \cong T^n$ in that case. $\square$

Remark 4.19. Everything in the discussion above Proposition 4.18 holds for both integrable and commuting systems. The proof of it shows that every regular fibre of a *commuting* system is diffeomorphic to $T^k \times \mathbb{R}^{n-k}$ for some $k \in \{0, \ldots, n\}$.

Definition 4.20. *Let $b \in B_{\mathrm{reg}}$ be a regular value of an integrable system $F\colon X \to B$. The set*

$$\Lambda_b^F = \mathrm{stab}(x) = \{\tau \in \mathbb{R}^n \mid \Phi^F|_{F^{-1}(b)} = \mathrm{id}\} \subset \mathbb{R}^n$$

for any $x \in F^{-1}(b)$ is called period lattice (of F) at b. *We call*

$$\Lambda^F = \{(\tau, b) \in \mathbb{R}^n \times B_{\mathrm{reg}} \mid \tau \in \Lambda_b\} \subset \mathbb{R}^n \times B_{\mathrm{reg}}$$

period lattice. *We often just write $\Lambda = \Lambda^F$ in case there is no ambiguity.*

Again, the first definition does not depend on the choice of $x \in F^{-1}(b)$. Proposition 4.15 shows that the set $\Lambda_b \subset \mathbb{R}^n$ really is a lattice. We think of $\Lambda \to B_{\mathrm{reg}}$ as a lattice bundle. Note that for now we have not shown that $b \mapsto \Lambda_b$ depends smoothly on $b \in B_{\mathrm{reg}}$, i.e. that this is an honest bundle. This follows from Proposition 4.23, see Remark 4.24.

Exercise 4.21 (*). Let $F\colon X \to \mathbb{R}^n$ be a commuting/integrable system and $U \subset B_{\mathrm{reg}}$. Let $f\colon U \to V = F(U) \subset \mathbb{R}^n$ be a diffeomorphism. Then the period lattice transforms as follows:

$$\Lambda_{f(b)}^{f\circ F} = (Df(b)^{-1})^T \Lambda_b^F. \tag{50}$$

4.5. **Symplectic covering.** The previous discussion has shown that, for any integrable system $F\colon X \to B$, the restriction

$$F|_{F^{-1}(B_{\mathrm{reg}})}\colon F^{-1}(B_{\mathrm{reg}}) \to B_{\mathrm{reg}}$$

is a T^n-bundle. This follows, via the fibration lemma, from the compactness of the fibres. Additionally, this fibre bundle comes with a fibre-wise $\mathbb{R}^n$-action and thus a trivialization corresponds to picking a section of the bundle. Thus every choice of smooth section gives a natural trivialization of the bundle.

Definition 4.22. *Let $U \subset B_{\mathrm{reg}}$. A* smooth section (of F over U) *is a smooth map $s\colon B_{\mathrm{reg}} \supset U \to F^{-1}(U)$ satisfying*

$$F \circ s = \mathrm{id}_U\,.$$

By differentiating this equation, we find any section automatically has maximal rank, meaning that it gives an embedding of U into X. Because of the local triviality of F, smooth sections exist locally. Given a local section, we can define an associated symplectic covering of the integrable system.

Proposition 4.23. *Let $U \subset B_{\mathrm{reg}}$ be a subset of the regular values of an integrable system $F\colon X \to B$ which admits a smooth section $s\colon U \to F^{-1}(U) \subset X$. Then the map*

$$P = P^F_{U,s}\colon \mathbb{R}^n \times U \to F^{-1}(U), \quad (\tau, b) \mapsto \Phi^F_{-\tau}(s(b))$$

(1) preserves fibres in the sense that $F(P(\tau,b)) = b$;
(2) is a local diffeomorphism satisfying

$$P^*\omega = \textstyle\sum_i db_i \wedge d\tau_i + \pi_U^*\beta,$$

where $\pi_U\colon \mathbb{R}^n \times U \to U$ is the obvious projection map and $\beta = s^*\omega$;
(3) is not injective. In fact it satisfies

$$P(\tau, b) = P(\tau', b') \quad \text{if and only if} \quad b = b' \text{ and } \tau' - \tau \in \Lambda_b,$$

where Λ_b denotes the period lattice of F at b as in Definition 4.20.

Proof. Point *(1)* is a straightforward computation,

$$F(P(\tau, b)) = F(\Phi^F_{-\tau}(s(b))) = F(s(b)) = b.$$

Let's prove *(2)*. Let us compute the push-forward under P of the standard basis vectors ∂_{τ_i} and ∂_{b_i}. As in the case of cotangent bundles, we make a natural abuse of notation by denoting by ∂_{b_i} both the vector fields on $\mathbb{R}^n \times U$ and their projections to U. Using (49) and the fact that the Hamiltonian flows ϕ^{F_i} are generated by the Hamiltonian vector fields X_{F_i}, we compute

$$P_*\partial_{\tau_i} = \frac{\partial(\phi^{F_1}_{-\tau_1} \circ \ldots \circ \phi^{F_n}_{-\tau_n})(s(b))}{\partial \tau_i} = -X_{F_i}.$$

On the other hand,

$$P_*\partial_{b_i} = (\Phi^F_{-\tau})_* s_* \partial_{b_i}.$$

The former set of vectors spans the tangent space $TF^{-1}(b)$ of the fibre and the latter spans (some Φ^F-translate of) the tangent space of the section s. Since these two subspaces are transversal, the differential of P has maximal rank and is thus a local diffeomorphism. To determine $P^*\omega$, compute

$$(P^*\omega)(\partial_{\tau_i}, \partial_{\tau_j}) = \omega(X_{F_i}, X_{F_j}) = \{F_i, F_j\} = 0,$$

and

$$(P^*\omega)(\partial_{b_i}, \partial_{b_j}) = \omega((\Phi^F_\tau)_* s_* \partial_{b_i}, (\Phi^F_\tau)_* s_* \partial_{b_j}) = \omega(s_*\partial_{b_i}, s_*\partial_{b_j}) = \beta(\partial_{b_i}, \partial_{b_j}),$$

where we have used that Φ^F_τ acts by symplectomorphisms for all $\tau \in \mathbb{R}^n$ and have set $\beta = s^*\omega$. Furthermore, we find

$$(P^*\omega)(\partial_{b_j}, \partial_{\tau_i}) = \omega((\Phi^F_\tau)_* s_* \partial_{b_j}, -X_{F_i}) = dF_i((\Phi^F_\tau)_* s_* \partial_{b_j}) = dF_i(s_*\partial_{b_j}) = db_i(\partial_{b_j}),$$

where we have used (in the same order) that F_i is invariant under the action Φ^F, the fact that $F \circ s = \mathrm{id}_U$, and that the coordinate b_i in $U \subset \mathbb{R}^n$ is the image of F_i. The expression $db_i(\partial_{b_j})$ is equal to 1 if $i = j$ and vanishes otherwise. This proves *(2)*.

For claim *(3)*, the *if* direction follows from the definition of P as $P(\tau, b) = \Phi^F_{-\tau}(s(b))$ and the fact that Λ_b is defined as the stabilizer the $\mathbb{R}^n$-action $\tau \mapsto \Phi^F_\tau$ on the fibre of F. For the *only if* direction, apply F and point *(1)* to find that $b = b'$. Furthermore, if $P(\tau, b) = P(\tau', b)$, then $\Phi^F_{\tau'-\tau}(s(b)) = s(b)$ meaning that $\tau' - \tau \in \Lambda_b$ by definition of Λ_b as the stabilizer. □

Remark 4.24. By Proposition 4.23 point *(3)*, we find that

$$\Lambda|_U = P^{-1}(\operatorname{graph} s) \subset \mathbb{R}^n \times U. \tag{51}$$

In particular $\Lambda|_U$ is smooth since P is a local diffeomorphism. Since smooth sections always exist locally, we deduce that Λ is a smooth submanifold.

4.6. **Lagrangian submanifolds and Lagrangian sections.** By Proposition 4.23, we can find a chart P for the system which satisfies

$$P^*\omega = \textstyle\sum_i db_i \wedge d\tau_i + \pi_U^*\beta, \quad \text{for } \beta = s^*\omega.$$

Note that the *error term* β for P to be a symplectomorphism with respect to $\sum_i db_i \wedge d\tau_i$ depends on the choice of section s. In fact, we can get rid of it by choosing a *Lagrangian* section, instead of a just a smooth one. Let us discuss Lagrangian submanifolds. These are of crucial importance in all of symplectic geometry.

Definition 4.25. *A submanifold $L \subset (X, \omega)$ with $2 \dim L = \dim X$ is called* Lagrangian *if the symplectic form vanishes on its tangent spaces, $\omega|_{TL} = 0$. An embedding $\iota \colon L \hookrightarrow X$ is called* Lagrangian *if $\iota^*\omega = 0$.*

Remark 4.26. Why do we care about Lagrangian submanifolds? They are ubiquitous in symplectic geometry/topology. Very often, they are responsible for symplectic rigidity. A more down-to-earth explanation related to dynamics and physics is the following one. Let $H \colon X \to \mathbb{R}$ be any autonomous Hamiltonian. We know that any level set $H^{-1}(c)$ is an invariant set of the Hamiltonian flow ϕ^F_t. *Any Lagrangian submanifold* contained in $H^{-1}(c)$ is automatically an invariant subset, too! Note that, generically, the level set has dimension $2n - 1$ and the Lagrangian has dimension n, meaning that having invariant Lagrangian subsets is much more useful to solve the Hamiltonian system.

Exercise 4.27. Prove this. To be precise: Prove that for any (autonomous) Hamiltonian $H \colon (X, \omega) \to \mathbb{R}$ and any Lagrangian $L \subset (X, \omega)$ satisfying $L \subset F^{-1}(c)$ for some $c \in \mathbb{R}$, the Lagrangian L is an invariant subset of the Hamiltonian flow ϕ^H_t.

The image of a Lagrangian embedding is a Lagrangian submanifold. Any embedding having a Lagrangian submanifold as its image is Lagrangian. We will often call Lagrangian submanifolds simply *Lagrangians*. This is not to be confused with the Lagrangian functionals from classical mechanics.

Examples 4.28. (1) Every smooth curve in a symplectic surface is a Lagrangian submanifold. This just follows from the skew-symmetry of symplectic forms.

(2) Let $(T^*Q, \omega_{\mathrm{can}})$. Then the zero-section and the fibres are Lagrangian submanifolds. Indeed, recall that locally we can write $\omega_{\mathrm{can}} = \sum_i dq_i \wedge dp_i$ and the zero-section is determined by $\{p = 0\}$ and the fibres by $\{q = q_*\}$ for some point $q_* \in Q$. Therefore all dp_i vanish on the tangent spaces on the former, whereas all dq_i vanish on the tangent spaces of the latter.

(3) Let $f \colon \mathbb{R}^n \to \mathbb{R}$ be a smooth function. Consider its gradient $x \mapsto \nabla f(x)$ as a map $\mathbb{R}^n \to \mathbb{R}^n$. Then the graph

$$\{(x, \nabla f(x)) \subset \mathbb{R}^n \times \mathbb{R}^n\}$$

is Lagrangian in $\mathbb{R}^{2n}$ equipped with the standard symplectic form $\sum_i dx_i \wedge dy_i$. One can check this statement by hand. However, we will see a generalization in Proposition 4.29.

(4) Let $F \colon X \to B$ be a commuting/integrable system. Every fibre $F^{-1}(b)$ is Lagrangian. Indeed, the tangent space $T_x F^{-1}(b)$ is spanned by the vectors $X_{F_1}|_x, \dots, X_{F_n}|_x$ and

$$\omega(X_{F_i}, X_{F_j}) = \{F_i, F_j\} = 0.$$

Proposition 4.29. *Let Q be a smooth manifold. The graph of a one-form $\alpha \in \Gamma(T^*Q) = \Omega^1(Q)$ is Lagrangian if and only if α is closed, $d\alpha = 0$. In particular, the graph*

$$\operatorname{graph} df = \{df|_q \in T^*Q \mid q \in Q\} \tag{52}$$

of any differential of a smooth function $f \in C^\infty(Q)$ is Lagrangian.

Proof. Let $\alpha \in \Omega^1(Q)$ and let $0_Q \colon Q \to T^*Q$ be the zero-section. Then the embedding $t_{-\alpha} \circ 0_Q \colon Q \to T^*Q$ has the graph of α as its image. Here $t_{-\alpha}$ denotes translation the fibrewise translation $\eta \mapsto \eta + \alpha_{\pi_Q(\eta)}$ as in Definition 2.22. By Proposition 2.23, we can compute

$$(t_{-\alpha} \circ 0_Q)^* \omega_{\mathrm{can}} = 0_Q^*(\omega_{\mathrm{can}} - \pi_Q^* d\alpha) = -d\alpha,$$

proving that the embedding (and thus its image) is Lagrangian if and only if $d\alpha$. Equivalently, one can use Proposition 2.27 point *(2)*. □

Lagrangians have the following remarkable property: Every two compact Lagrangians which are diffeomorphic have symplectomorphic neighbourhoods. For a fixed diffeomorphism type L, one often takes the zero-section in the cotangent bundle T^*L as model to state this result as follows:

Theorem 4.30 (Weinstein's Lagrangian neighbourhood theorem). *Let $L \subset (X, \omega)$ be a compact Lagrangian submanifold. Then there is a neighbourhood W of L and a symplectomorphism*

$$\chi \colon W \to \chi(W) \subset (T^*L, \omega_{\mathrm{can}}), \quad \chi(L) = 0_L,$$

where $(T^*L, \omega_{\mathrm{can}})$ denotes the cotangent bundle of L equipped with the canonical symplectic form ω_{can}, which we have discussed in §2.4 and 0_L denotes its zero-section.

This result follows from a variant of the so-called Moser's trick, we refer to [2, Chapter 8] or [4, §3.4] for details.

Let us return to sections of commuting/integrable systems.

Definition 4.31. *Let $F \colon X \to B$ be a commuting/integrable system and $U \subset B_{\mathrm{reg}}$ be an open subset of regular values. A* Lagrangian section (of F over U) *is a smooth section $s \colon U \to F^{-1}(U)$ satisfying $s^*\omega = 0$.*

Note that this means that s is a Lagrangian embedding of U into $F^{-1}(U)$ and thus that the graph of s is a Lagrangian submanifold of (X, ω). A crucial ingredient in the proof of the Arnol'd–Liouville theorem is the local existence of Lagrangian sections.

Lemma 4.32. *Let $b \in B_{\mathrm{reg}}$ in the basis of an integrable system $F \colon X \to B$. Then there is a neighbourhood $U \subset B_{\mathrm{reg}}$ of b and a Lagrangian section $s \colon U \to F^{-1}(U)$.*

Proof. The fibre $L = F^{-1}(b)$ is compact by our definition of integrable systems, it is a submanifold since $b \in B_{\mathrm{reg}}$ and it is Lagrangian by Example 4.28 (4). Hence, Theorem 4.30 yields a symplectomorphism χ of a neighbourhood W of L to a neighbourhood $\chi(W) \subset T^*L$ of the zero section $0_L \subset T^*L$. Pick a point $y \in L$ and let $S = \chi^{-1}(T^*_y L \cap \chi(W))$ be the inverse image of a cotangent fibre under χ. Note that S is Lagrangian since any cotangent fibre is Lagrangian and χ is a symplectomorphism. Let us now construct a Lagrangian section having S as its image. Since $T^*_y L$ is transverse to 0_L, its inverse image S is transverse to L,

$$T_x X = T_x S \oplus T_x L = T_x S \oplus T_x F^{-1}(b).$$

Since F is a submersion on a neighbourhood of L and $T_x F^{-1}(b) = \ker DF|_x$, we find that $DF|_x$ restricts to an isomorphism $T_x S \to T_b B$ for dimensional reasons. Therefore, there is a neighbourhood U of b such that

$$F|_S \colon S \cap F^{-1}(U) \to U$$

is a diffeomorphism. Let $s = F|_S^{-1}$. This is the desired Lagrangian section. Indeed, it satisfies $F \circ s = \mathrm{id}_U$ by definition, and since S is a Lagrangian submanifold, we have $s^*\omega = 0$. □

Corollary 4.33. *The period lattice $\Lambda^F \subset \mathbb{R}^n \times B_{\mathrm{reg}}$ is a* Lagrangian *submanifold with respect to the standard symplectic form $\sum_i db_i \wedge d\tau_i$.*

Exercise 4.34 (*). Prove this.

Remark 4.35. The same local existence statement holds for *commuting systems*. The proof needs to be slightly modified, since the fibre may not be compact in that case, meaning that Theorem 4.30 does not apply. One can use Darboux's theorem instead (see Theorem 2.35) to make a similar argument. See for example [3, Exercise 1.48].

4.7. **Proof of the Arnol'd–Liouville theorem.** *Proof of Theorem 4.8.* Let $b_0 \subset B_{\mathrm{reg}}$ be regular point and $U \subset B_{\mathrm{reg}}$ a neighbourhood of that point which

(1) is simply connected, $\pi_1(U) = 0$;
(2) admits a Lagrangian section $s \colon U \to F^{-1}(U)$.

This is possible by Lemma 4.32. The proof has two steps.

Step 1: By Proposition 4.23, there is a smooth covering $P = P_{U,s} \colon \mathbb{R}^n \times U \to F^{-1}(U)$ which is symplectic with respect to the standard symplectic form,

$$P^*\omega = \textstyle\sum_i db_i \wedge d\tau_i.$$

The fibre of P over any $s(b) \in X$ is Λ_b. Therefore, it is very natural to suspect that $F^{-1}(U)$ is symplectomorphic to some kind of quotient of $\mathbb{R}^n \times U$ by the period lattice $\Lambda|_U$. Indeed, the first step is proving that P induces a fibre-preserving symplectomorphism

$$((\mathbb{R}^n \times U)/\Lambda|_U, \textstyle\sum_i db_i \wedge d\tau_i) \cong (F^{-1}(U), \omega). \tag{53}$$

We first need to make sense of the equivalence relation in (53). We now make two conceptual shifts. On one hand, we can consider

$$(\mathbb{R}^n \times U, \textstyle\sum_i db_i \wedge d\tau_i) = (T^*U, \omega_{\mathrm{can}}),$$

via the usual identification (13) under which $b_i = q_i$ and $\tau_i = p_i$ in the notation of §2.3. For consistency with the rest of this section, we will not switch to the q_i, p_i-notation, but instead keep using b_i, τ_i.

On the other hand, we view $\Lambda \to B_{\mathrm{reg}}$ as a covering map. Since U is simply connected, the restriction $\Lambda|_U \to U$ is a trivial covering. In particular, we can pick smooth sections $\alpha_1, \dots, \alpha_n \colon U \to T^*U$ trivializing the bundle, i.e. for which we have

$$\Lambda_b = \mathrm{span}_{\mathbb{Z}}\{\alpha_i(b)\}, \quad \text{for all } b \in U.$$

The α_i are differential one-forms. By Corollary 4.33, the period lattice is Lagrangian and thus, by Proposition 4.29, the one-forms α_i are closed. This allows us to define the $\mathbb{Z}^n$-action

$$A \colon \mathbb{Z}^n \times T^*U \to T^*U, \quad (k, (\tau, b)) \mapsto A_k(\tau, b) = (\tau - \textstyle\sum_i k_i \alpha_i(b), b).$$

This action is

(1) properly discontinuous (for the quotient to be a smooth manifold in the first place);
(2) symplectic in the sense that $A_k^*(\sum_i db_i \wedge d\tau_i) = \sum_i db_i \wedge d\tau_i$.

Exercise 4.36. Prove this. *Hint:* The proof of the second claim uses something we already know and is one line.

The orbit of the zero-section under this action is all of $\Lambda|_U$. Therefore, we can define the quotient $T^*U/\Lambda|_U$ as T^*U/A, to get a well-defined symplectic manifold. Recall that Λ_b is by definition the stabilizer of the action Φ^F on the fibre $F^{-1}(b)$. Furthermore $\alpha_i(b) \in \Lambda_b$. Therefore we find

$$P(A_k(\tau, b)) = \Phi^F_{(-\sum_i k_i\alpha_i(b))}(P(\tau, b)) = P(\tau, b),$$

and P descends to a map $T^*U/\Lambda_U \to F^{-1}(U)$. By Proposition 4.23 it is a symplectomorphism. This finishes step 1.

Remark 4.37. Let us discuss where we are so far. The symplectomorphism $T^*U/\Lambda_U \to F^{-1}(U)$ induced by P looks already very much like the (the inverse of the) symplectomorphism ψ that we look for in the statement of the theorem. Indeed, we have a commutative diagram

$$\begin{tikzcd} F^{-1}(U) \arrow[r, "P^{-1}"] \arrow[d, "F"'] & T^*U/\Lambda|_U \arrow[dl] \\ U & \end{tikzcd}$$

where P^{-1} is a symplectomorphism and $T^*U/\Lambda|_U$ is a torus bundle with Lagrangian fibres! However, the bundle $T^*U/\Lambda|_U$ is not the *standard* torus bundle $J\colon \mathcal{T}(U) \to U$. Indeed, the lattice $\Lambda|_U$ may not be standard, meaning that the associated Hamiltonian flow may not be periodic (and, even if they are, not of constant unit period). If $\Lambda|_U$ was *standard*, meaning $\Lambda|_U = \mathbb{Z}^n \times U$, then the theorem would be proven. Note that there is another difference between what we have proven so far and the actual statement. Indeed, no diffeomorphism $f\colon U \to V \subset \mathbb{R}^n$ of the base has intervened so far. In fact, those two things go hand in hand: We will use the freedom provided by the choice of f to *make the lattice standard*. We know from Exercise 4.21 that $\Lambda^{f\circ F} = (Df^{-1})^T\Lambda^F$. Therefore, making the lattice standard means that we need to solve $\mathbb{Z}^n = (Df(b)^{-1})^T\Lambda^F_b$ for all b to find a diffeomorphism f. That such a solution exists is far from obvious: Indeed, it corresponds to *integrating* a family of prescribed differentials $b \mapsto Df(b)$ to a diffeomorphism $f\colon U \to V$. That's the second step of the proof.

Step 2: We again denote by $\alpha_i \in \Omega^1_{\mathrm{cl}}(U)$ the closed-one forms spanning $\Lambda|_U$. Using (for the second time) the fact that U is simply connected, we find that $H_1(U)$ (as the abelianization of $\pi_1(U)$) is trivial and hence that

$$0 = H^1(U;\mathbb{R}) = H^1_{\text{de Rham}}(U) = \frac{\ker(d\colon \Omega^1(U) \to \Omega^2(U))}{\operatorname{im}(d\colon C^\infty(U) \to \Omega^1(U))}.$$

In particular, we can find

$$\alpha_i = df_i, \quad \text{for some } f_i \in C^\infty(U).$$

Let us now set

$$f = (f_1, \ldots, f_n)\colon U \to V = \operatorname{im} f \subset \mathbb{R}^n. \tag{54}$$

We claim that this is the diffeomorphism in the statement of Theorem 4.8. First, it is a local diffeomorphism since $df_i = \alpha_i$ and the α_i are linearly independent since they form a basis of the lattice. Up to picking a smaller U, the map f is thus a diffeomorphism.

Second, by Proposition 4.5, the map $f \circ F$ is again an integrable system and by Exercise 4.21, we have

$$\Lambda^{f\circ F} = (Df^T)^{-1}\Lambda^F.$$

But $Df^T = (df_1, \dots, df_n) = (\alpha_1, \dots, \alpha_n)$ and therefore we find $Df^T e_i = \alpha_i$ and $(Df^T)^{-1}\alpha_i = e_i$. Since the α_i span Λ^F, we find that $\Lambda^{f\circ F} = \mathbb{Z}^n \times V$ is the trivial lattice. To conclude the proof, it suffices to apply the result in Step 1 to the integrable system $f \circ F$ equipped with the Lagrangian section $s \circ f^{-1}$. Indeed, we then obtain a symplectomorphism

$$P = P^{f\circ F}_{V,s\circ f^{-1}} \colon (\mathbb{R}^n \times V)/\Lambda^{f\circ F} \to (f\circ F)^{-1}(V) = F^{-1}(U)$$

such that $f \circ F \circ P$ agrees with the obvious projection to V. Note that the since $\Lambda^{f\circ F} = \mathbb{Z}^n \times V$, we have

$$(\mathbb{R}^n \times V)/\Lambda^{f\circ F} = (\mathbb{R}^n \times V)/(\mathbb{Z}^n \times V) = \mathcal{T}(V).$$

The obvious projection to V coincides with the integrable system $J \colon \mathcal{T}(V) \to V$. Setting $\psi = P^{-1}$ yields the claim. □

4.8. **Discussion: Compactness of fibres.** Why do we need to assume that the fibres of F are compact[3]? Can we expect an Arnol'd–Liouville type theorem if we drop the compactness assumption? On one hand, part of what we have discussed holds for commuting systems (i.e. without compactness of the fibres), too: The fundamental $\mathbb{R}^n$-action on fibres exists, the period lattice is a lattice,... On the other hand, some parts fail spectacularly: The rank of the lattice does not need to be constant, even away from critical values! Or in other words, there exist commuting systems which have regular values $b \in B_{\mathrm{reg}}$ which do not have neighbourhoods over which F is a fibration. In that case there cannot be an Arnol'd–Liouville type theorem. On a deeper level, this follows from a failure of the fibration lemma for non-compact fibres.

Theorem 4.38 (Fibration lemma)**.** *Let $f \colon X \to Y$ be a smooth map restricting to a proper submersion $f|_{f^{-1}(U)} \colon f^{-1}(U) \to U$ for some $U \subset Y$. Fix $y_0 \in U$. There is a diffeomorphism $\Psi \colon U \times f^{-1}(y_0) \to f^{-1}(U)$ such that $f \circ \Psi$ is equal to the obvious projection to U.*

Here is an example where the fibration lemma fails when properness (i.e. compact fibres) is dropped.

[3]Many thanks to Esther for asking this question.

Example 4.39. Consider the function[4]

$$F\colon \mathbb{R}^2 \to \mathbb{R}, \quad F(x,y) = \frac{2x}{1+x^2+y^2}.$$

Let us interpret this geometrically: It is the first component of the inverse to the stereographic projection. In other words, it is the function one obtains by mapping $\mathbb{R}^2$ to the complement of the north pole in S^2 and then composing with the projection to a coordinate. Therefore all level sets are diffeomorphic to the level sets of a coordinate function on the sphere, except for the one containing the north pole. This shows that $F^{-1}(b)$ is diffeomorphic to:

(1) the empty set for $|b| > 1$;
(2) to a point for $b \in \{-1, 1\}$;
(3) to a circle for $b \in (-1,1) \setminus \{0\}$;
(4) to $\mathbb{R}$ for $b = 0$.

The value $b = 0$ is regular, but obviously neighbouring level sets are not diffeomorphic to $F^{-1}(0)$. This violates the fibration lemma.

The previous example can be viewed a commuting system $F\colon (\mathbb{R}^2, \omega_0) \to \mathbb{R}$. There cannot be a Arnol'd–Liouville type theorem for $b = 0$, since the fibration is not locally trivial. Let us now discuss what happens to period lattices in this example. Let $T\colon (-1,1) \setminus \{0\}$ be the period function defined in Exercise 4.11, meaning that the flow on the level set $F^{-1}(b)$ is $T(b)$-periodic. In Proposition 4.12, we saw that $T(b)$ is proportional to the symplectic area bounded by $F^{-1}(b)$. In this example we find that the period lattices are

$$\Lambda_0 = \{0\}, \quad \Lambda_b = T(b)\mathbb{Z}, \quad \text{for } \lim_{b \to 0} T(b) = \infty.$$

However, the following can be shown, see [3, Lemma 1.34].

Proposition 4.40. *Let $b_0 \in B_{\mathrm{reg}}$ for a commuting system $F\colon X \to B$. Then there is a neighbourhood $U \subset B_{\mathrm{reg}}$ of b_0 such that*

$$\operatorname{rk} \Lambda_b \geqslant \operatorname{rk} \Lambda_{b_0}, \quad \text{for all } b \in U.$$

In other words, the rank is lower semi-continuous as a function of the base point b. In conclusion, if one would like for an Arnol'd–Liouville type theorem to hold in the context of commuting systems, one needs to at least assume that the rank function $b \mapsto \Lambda_b$ is locally constant.

[4]Thanks to GPT 5.4 which came up with this example. The geometric interpretation was later added by myself.

References

[1] V. I. Arnold. *Mathematical methods of classical mechanics.* Springer-Verlag, New York, 1978. Translated from the Russian by K. Vogtmann and A. Weinstein, Graduate Texts in Mathematics, 60.
[2] A. Cannas da Silva. *Lectures on symplectic geometry*, volume 1764 of *Lecture Notes in Mathematics.* Springer-Verlag, Berlin, 2001.
[3] J. D. Evans. Lectures on lagrangian torus fibrations, 2021.
[4] D. McDuff and D. Salamon. *Introduction to symplectic topology.* Oxford Graduate Texts in Mathematics. Oxford University Press, Oxford, third edition, 2017.
[5] L. Polterovich. *The geometry of the group of symplectic diffeomorphisms.* Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 2001.

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