

INTEGRABLE SYSTEMS AND SYMPLECTIC GEOMETRY

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ABSTRACT.

1. INTRODUCTION

Recommended exercises are marked by (*).

Difficult exercises are marked by (!).

Text appearing in light grey has NOT been discussed in the lecture and is therefore not exam material.

2. MECHANICS AND SYMPLECTIC GEOMETRY

2.1. **Hamilton's equations.** Here is Newton's equation of motion from classical mechanics,

$$\ddot{q}(t) = -\nabla V(q) \tag{1}$$

describing the position depending on time $t \mapsto q(t) \in \mathbb{R}^n$ of a particle of unit mass submitted to a force field of a potential $V: \mathbb{R}^n \rightarrow \mathbb{R}$. Here we take the potential to not depend on time, nor on the velocity $\dot{q}(t)$. This is an second order ODE in \mathbb{R}^n .

Hamilton's reformulation of that equation goes as follows: Set $p(t) = \dot{q}(t) \in \mathbb{R}^n$ and view it as another variable of the system, called *momentum*. This yields the equation of motion $\dot{p}(t) = -\nabla V(q)$ coupled with $\dot{q}(t) = p(t)$ Setting

$$H(q, p) = \frac{\|p\|^2}{2} + V(q) \tag{2}$$

for Euclidean norm $\|p\|^2 = p_1^2 + \dots + p_n^2$ allow us to write Newton's equation as a system of coupled first order ODEs

$$\dot{q}_i(t) = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i(t) = -\frac{\partial H}{\partial q_i}. \tag{3}$$

These are called *Hamilton's equations* and H is called *Hamiltonian (function)*. Physically speaking, H measures the total energy of the system, and is composed of the *kinetic energy*

$\frac{\|p\|^2}{2}$ and the *potential energy* $V(q)$. It is easy to check that the solutions preserve energy, meaning that the function $t \mapsto H(q(t), p(t))$ is constant in t ,

$$\frac{dH}{dt} = \sum_{i=1}^n \frac{\partial H}{\partial q_i} \dot{p}_i + \frac{\partial H}{\partial p_i} \dot{q}_i = 0,$$

where we have used (3).

Example 2.1 (Harmonic oscillator). Let $n = 1$ and $V(q) = \alpha q^2$ for some $\alpha > 0$. Then (3) describe the motion of a unit mass attached to a spring which is at rest at $q = 0$. The parameter α has something to do with the stiffness of the spring. To say something about the harmonic oscillator, note that the total energy is preserved, i.e. the function $t \mapsto H(q(t), p(t))$ is constant for all curves solving (3). This allows us to say something very strong about the solutions of these equations, namely that they are constrained to the ellipses $\{\frac{p^2}{2} + \alpha q^2 = c\}$. We can draw the so-called *phase portrait* of the Harmonic oscillator, see ???. Of course, we could just have written down explicit solutions to the Harmonic oscillator instead, but closed formulae for this kind of problem exist only in very rare cases. The kind of geometric arguments above will inspire much of the rest of this lecture.

2.2. Symplectic geometry. The natural arena to set up a problem of classical mechanics is a *symplectic manifold*.

Definition 2.2. A symplectic manifold is a pair (X^{2n}, ω) consisting of a smooth even-dimensional manifold X and a differential two form which is closed and non-degenerate.

By *closed*, we mean that its exterior differential vanishes, i.e. $d\omega = 0$. By *non-degenerate* we mean the assignment $Z \mapsto \iota(Z)\omega = \omega(Z, -)$ defines a bundle isomorphism $TX \rightarrow T^*X$. Non-degeneracy allows us to turn any smooth function $H \in C^\infty(X)$ into a vector field X_H by setting

$$dH = \iota(X_H)\omega. \tag{4}$$

Definition 2.3. In this context, we call $H \in C^\infty(X)$ a Hamiltonian (function), we call $X_H \in \Gamma(TX)$ Hamiltonian vector field, and we call the flow $t \mapsto \phi_t^H$ it generates Hamiltonian flow. The triple (X, ω, H) is called Hamiltonian system.

By *the flow it generates* we mean the one-parametric family of diffeomorphisms ϕ_t^H satisfying $\frac{d\phi_t^H}{dt} = X_H$ for all times t . **All throughout, we assume that Hamiltonian flows are defined for all times.** The Hamiltonian flow of H preserves H and the symplectic form ω ,

$$H \circ \phi_t^H = H, \quad (\phi_t^H)^*\omega = \omega. \tag{5}$$

Indeed, differentiating with respect to time, we obtain, in the first case

$$\frac{d}{dt}(H \circ \phi_t^H) = dH(X_H) = (\iota(X_H)\omega)(X_H) = \omega(X_H, X_H) = 0.$$

The second equality is (4) and the last one follows from the skew-symmetry $\omega(X, Y) = -\omega(Y, X)$ of differential two forms. In the second case we obtain

$$\frac{d}{dt}(\phi_t^H)^*\omega = \mathcal{L}_{X_H}\omega = d\iota(X_H)\omega + \iota(X_H)d\omega = ddH + 0 = 0,$$

where \mathcal{L}_Y is the *Lie derivative* of tensor fields along a vector field Y . In case the tensor field is a differential form $\alpha \in \Omega^1(X)$, it can be computed by the co-called *Cartan's magic formula* $\mathcal{L}_Y\alpha = d\iota(Y)\alpha + \iota(Y)d\alpha$, which is what we have used to find the second equality.

Example 2.4. Let $\mathbb{R}^2 = \{(q, p)\}$ be equipped with the differential two form

$$\omega_0 = dq \wedge dp.$$

This is just the standard area form. It is closed, $d\omega_0 = ddx \wedge dy + dx \wedge ddy = 0$ (every differential form in the top degree is automatically closed anyways) and furthermore we have

$$\iota(\partial_q)\omega_0 = dp, \quad \iota(\partial_p)\omega_0 = -\iota(\partial_p)dp \wedge dq = -dq \quad (6)$$

where $\partial_q, \partial_p \in T\mathbb{R}^2$ denote the vector fields pointing in the coordinate directions. This implies non-degeneracy, since ∂_q, ∂_p span $T\mathbb{R}^2$ and dq, dp span $T^*\mathbb{R}^2$. Let us now look at some Hamiltonians:

$$H_1(q, p) = p, \quad H_2(q, p) = q^2 + p^2.$$

(4) yields

$$X_{H_1} = \partial_q, \quad X_{H_2} = 2p\partial_q - 2q\partial_p.$$

Indeed, (6) shows that we can go from dH to X_H by replacing every dp by ∂_q and every dq by $-\partial_p$. The first Hamiltonian flow acts by translation in the q -direction and the second one by rotation.

The product of two symplectic manifolds is again symplectic.

Exercise 2.5. Let $\mathbb{R}^4 = \{(q_1, p_1, q_2, p_2)\}$ be equipped with $\omega_0 = dq_1 \wedge dp_1 + dq_2 \wedge dp_2$. The Hamiltonian

$$G(q_1, p_1, q_2, p_2) = p_1q_2 - p_2q_1$$

is sometimes called *angular momentum*. Compute its flow ϕ_t^G . How does this flow act on the (q_1, q_2) - and on the (p_1, p_2) -planes?

For any n , denote the coordinates on \mathbb{R}^{2n} by $q_1, \dots, q_n, p_1, \dots, p_n$ and let

$$\omega_0 = \sum_{i=1}^n dq_i \wedge dp_i.$$

This is called the *standard symplectic form*. Let H be a Hamiltonian. Its differential is given by

$$dH = \sum_{i=1}^n \left(\frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial p_i} dp_i \right),$$

and thus

$$X_H = \sum_{i=1}^n \left(\frac{\partial H}{\partial p_i} \partial_{q_i} - \frac{\partial H}{\partial q_i} \partial_{p_i} \right)$$

In other words, the system $(\dot{q}(t), \dot{p}(t)) = X_H$ exactly corresponds to Hamilton's equations (3)! This shows that we have generalized them to manifolds.

Definition 2.6. We call (4) Hamilton's equation (*singular!*) from now on.

This means that the first equation in (5) is a generalized form of energy conservation. As for the second equation, let us make the following historical remark:

Remark 2.7. Liouville noticed in the 19th century that phase space of systems from classical mechanics carries a natural measure (or volume form) which is given, in so-called canonical coordinates, by

$$\text{vol} = dx_1 \wedge dy_1 \wedge \dots \wedge dx_n \wedge dy_n,$$

which is preserved by the mechanical system. The fact that the symplectic form ω is preserved is often (for example in [2]) attributed to Arnol'd. It is a strict refinement of Liouville's theorem. Indeed, we can write

$$n! \text{vol} = \omega_0^{\wedge n}$$

for the standard symplectic form ω_0 . Since $(\phi_t^H)^* \omega_0 = \omega_0$ by (5), we deduce that $(\phi_t^H)^* \text{vol} = \text{vol}$. This refinement has spectacular consequences! Indeed, symplectic topology studies different so-called *symplectic rigidity* phenomena. Those are properties exhibited by symplectic and Hamiltonian flows which volume-preserving flows do not have.

Definition 2.8. A diffeomorphism $\psi \in \text{Diff}(X)$ of a symplectic manifold (X, ω) is called symplectomorphism if $\psi^* \omega = \omega$. The set of these forms a subgroup

$$\text{Symp}(X, \omega) \subset \text{Diff}(X).$$

As we have shown in (5), every Hamiltonian flow is a symplectomorphism. Furthermore, conjugating a Hamiltonian flow by a symplectomorphism yields another Hamiltonian flow:

$$\psi \circ \phi_t^H \circ \psi^{-1} = \phi_t^{H \circ \psi^{-1}} \tag{7}$$

for all $H \in C^\infty(X)$ and all $\psi \in \text{Symp}(X, \omega)$. In other words, conjugation by a symplectomorphism corresponds to pulling back its Hamiltonian function. This can be proved by differentiating with respect to t , which yields $\psi_* X_H = X_{H \circ \psi^{-1}}$. This latter equation can be

proved by computing

$$\begin{aligned}
\iota(\psi_* X_H)\omega &= \omega(\psi_* X_H, \cdot) \\
&= \omega(\psi_* X_H, \psi_* \psi_*^{-1}(\cdot)) \\
&= \psi^* \omega(X_H, \psi_*^{-1}(\cdot)) \\
&= \omega(X_H, \psi_*^{-1}(\cdot)) \\
&= (\iota(X_H)\omega)(\psi_*^{-1}(\cdot)) \\
&= dH(\psi_*^{-1}(\cdot)) \\
&= d(H \circ \psi^{-1})(\cdot),
\end{aligned}$$

proving the claim by Hamilton's equation. In the fourth line, we have crucially used that ψ is a symplectomorphism, $\psi^* \omega = \omega$.

It is natural to ask now whether Hamiltonian flows are a group, too. This is correct, provided we allow for the Hamiltonians to be time-dependent! A *time-dependent Hamiltonian* is given by a function

$$H: [0, 1] \times X \rightarrow \mathbb{R}, \quad H(t, x) = H_t(x).$$

We can define X_H^t by $dH_t = \iota(X_H^t)\omega$ for every t . We still denote the corresponding flow by ϕ_t^H .

Definition 2.9. A diffeomorphism of X is called Hamiltonian diffeomorphism if it is the time-one map ϕ_1^H of a (possibly time-dependent) Hamiltonian H .

Although this is not immediate, it can be shown that these objects form a group, denoted by $\text{Ham}(X, \omega)$. See [2, Section 1.4] for details and proofs.

We have inclusions

$$\text{Ham}(X, \omega) \subset \text{Symp}(X, \omega) \subset \text{Diff}(X). \quad (8)$$

One can wonder how big the differences are between these groups. Let us briefly discuss the case of the inclusion $\text{Ham}(X, \omega) \subset \text{Symp}(X, \omega)$.

Proposition 2.10. For any Hamiltonian flow ϕ_t^H and any smooth $\gamma: S^1 \rightarrow X$, we have

$$\int_{[0,1] \times S^1} C_\gamma^* \omega = 0, \quad \text{where } C_\gamma: [0, 1] \times S^1 \rightarrow X, \quad C_\gamma(s, t) = \phi_s^H(\gamma(t)). \quad (9)$$

Geometrically, this means that any cylinder swept out by a curve under a Hamiltonian flow has vanishing symplectic area.

Proof. Compute

$$\frac{\partial C_\gamma}{\partial s} = X_s^H(\phi_s^H(\gamma(t))), \quad \frac{\partial C_\gamma}{\partial t} = (\phi_s^H)_* \dot{\gamma}(t),$$

and plug it in

$$\begin{aligned}
\int_{[0,1] \times S^1} C_\gamma^* \omega &= \int_{[0,1]} \int_{S^1} \omega(X_s^H(\phi_s^H(\gamma(t))), (\phi_s^H)_* \dot{\gamma}(t)) dt ds \\
&= \int_{[0,1]} \int_{S^1} dH_s((\phi_s^H)_* \dot{\gamma}(t)) dt ds \\
&= \int_{[0,1]} \int_{S^1} \frac{(H_s \circ \phi_s^H \circ \gamma)(t)}{dt} dt ds = 0
\end{aligned}$$

In the second line, we have used Hamilton's equation and in the last line the fundamental theorem of calculus combined with the fact that γ is defined on S^1 . \square

Using this, we can construct an example of a non-Hamiltonian symplectomorphism.

Example 2.11. Let $X = S^1 \times \mathbb{R}$, where we view $S^1 = \mathbb{R}/\mathbb{Z}$. The symplectic form $\omega_0 = dq \wedge dp$ on \mathbb{R}^2 is translation-invariant. Therefore, it descends to a well-defined form ω on X . Note that this form is exact, i.e. it satisfies

$$\omega = d(-pdq). \quad (10)$$

Now let $\psi_r(q, p) = (q, p + r)$ be a translation in the p -direction on X by an amount r . This is a symplectomorphism for all $r \in \mathbb{R}$. However we will see that it is Hamiltonian only if $r = 0$. Define the curve

$$\gamma: S^1 \rightarrow X, \quad \gamma(t) = (t, 0).$$

Suppose that ψ_r is a Hamiltonian diffeomorphism, i.e. there is ϕ_t^H with $\psi_r = \phi_t^H$. Then C_γ would have vanishing symplectic area by Proposition 2.10. However, by (10) and Stokes, we can compute

$$\int_{[0,1] \times S^1} C_\gamma^* d(-pdq) = \int_{[0,1] \times S^1} dC_\gamma^*(-pdq) = - \int_{\gamma(S^1)} pdq + \int_{\psi_r(\gamma(S^1))} pdq = r.$$

Since $\gamma(S^1) = S^1 \times \{p = 0\}$ and $\psi_r(\gamma(S^1)) = S^1 \times \{p = r\}$, we find that this is equal to $0 + r = r$. This proves that ψ_r is not Hamiltonian whenever $r \neq 0$. Let us point out that it is not enough to use that the cylinder bounding the circles $S^1 \times \{p = 0\}$ and $S^1 \times \{p = r\}$ has non-zero symplectic area! Indeed the Hamiltonian flow ϕ_t^H ending on ψ_r may be much more complicated than just the obvious translation $\phi_t^H = \psi_{tr}$.

Somewhat miraculously, there is a converse to Proposition 2.10, due to Banyaga.

Theorem 2.12. *Let (X, ω) be compact and connected and let $s \mapsto \psi_s \in \text{Symp}(X, \omega)$ be a smooth path of symplectomorphisms starting at $\psi_0 = \text{id}$. If the area (9) (now swept out by the symplectic family $s \mapsto \psi_s$) vanishes for all $\gamma: S^1 \rightarrow X$, then $\psi_1 \in \text{Ham}(X, \omega)$.*

Proving this is outside the scope of this lecture. We refer to [1, Chapter 10], and Theorem 10.2.5, in particular. Note that checking this condition on *all maps* $\gamma: S^1 \rightarrow X$ is not

convenient. Instead, one can restrict one's attention to homological data to obtain the so-called *flux map* which has values in $H^1(X; \mathbb{R})$. We will encounter its Lagrangian cousin later in this lecture, we refer again to [1] or to [2, Chapter 14] for details.

From the above discussion, it follows in particular that if $H^1(X; \mathbb{R}) = 0$, then any path of symplectomorphisms ψ_t is automatically Hamiltonian. This fact is much more elementary. To see this, let $Y_t \in \Gamma(TX)$ be the family of vector fields satisfying $\frac{d\psi_t}{dt} = Y_t$. In other words, Y_t is the *velocity vector* (in some infinite-dimensional space...) of the curve $t \mapsto \psi_t$. Since $\psi_t^* \omega = \omega$, we obtain $\psi_t^*(\mathcal{L}_{Y_t} \omega) = 0$, which implies, by Cartan's magic formula, that

$$0 = d\iota(Y_t)\omega + \iota(Y_t)d\omega = d\iota(Y_t)\omega, \quad (11)$$

meaning that the one-form $\iota(Y_t)\omega$ is closed for all t . Since $H^1(X; \mathbb{R}) = 0$, de Rham's theorem implies that every $\iota(Y_t)\omega$ has a primitive function H_t , meaning

$$dH_t = \iota(Y_t)\omega.$$

But this is exactly Hamilton's equation (4). Up to checking that the t dependence in $H_t = H(t, \cdot)$ is smooth (this is actually somewhat subtle), this proves the claim.

Remark 2.13. Let us briefly comment on de Rham's theorem and how it is used here. It says that the *de Rham cohomology group* $H_{\text{dR}}^1(X)$ is isomorphic to the usual cohomology $H^1(X; \mathbb{R})$. However de Rham cohomology precisely measures by how much a closed form fails to be exact, i.e.

$$H_{\text{dR}}^1(X) = \frac{\ker(d: \Omega^1(X) \rightarrow \Omega^2(X))}{\text{im}(d: C^\infty(X) \rightarrow \Omega^1(X))} = \frac{\Omega_{\text{cl}}^1(X)}{\Omega_{\text{ex}}^1(X)}.$$

Therefore, every closed one form has a primitive function whenever $H^1(X; \mathbb{R}) = 0$.

Remark 2.14. It is sometimes useful to view the spaces appearing in (8) as infinite-dimensional Lie groups. The Lie algebra of $\text{Diff}(X)$ is given by smooth vector fields. Indeed, differentiating a family of diffeomorphisms yields a vector field as *tangent vector*. The bracket on this Lie algebra is the bracket of vector fields (which is also called Lie bracket),

$$\text{Lie}(\text{Diff}(X)) = (\Gamma(TX), [\cdot, \cdot]).$$

The discussion surrounding (11) shows that a vector field Y is tangent to $\text{Symp}(X, \omega) \subset \text{Diff}(X)$ if and only if the form $\iota(Y)\omega$ is closed and tangent to $\text{Ham}(X, \omega)$ if and only if it is exact. In other words, we find $\text{Lie}(\text{Symp}(X, \omega)) \cong \Omega_{\text{cl}}^1(X)$ and $\cong \Omega_{\text{ex}}^1(X)$ in the Hamiltonian case. But the set of exact one-forms on X is precisely the set of functions on the manifold up to adding a constant to the function. Later we will define the Poisson bracket $\{\cdot, \cdot\}$ on smooth functions, which is exactly the Lie bracket on the Lie algebra of $\text{Ham}(X, \omega)$,

$$\text{Lie}(\text{Ham}(X, \omega)) = (C_0^\infty(X), \{\cdot, \cdot\}).$$

Here $C_0^\infty(X)$ denotes smooth functions up to adding a constant.

2.3. The cotangent bundle. The cotangent bundle $\pi_Q: T^*Q \rightarrow Q$ of any smooth manifold Q comes equipped with a *canonical symplectic form* ω_{can} . The so-obtained symplectic manifold

$$(T^*Q, \omega_{\text{can}})$$

is one of the quintessential examples in symplectic geometry. For us it will be crucial, because:

- (1) Classical mechanical systems on a smooth manifold Q correspond to a Hamiltonian system on T^*Q . The cotangent bundle is then called *phase space*.
- (2) The cotangent bundle (equipped with ω_{can} or a magnetic symplectic form) will serve as a symplectic model space for Lagrangian submersions, which we will study later on.

The symplectic form ω_{can} is exact, meaning that there is a one-form λ_{can} on T^*Q such that $\omega_{\text{can}} = -d\lambda_{\text{can}}$. There are multiple ways to define λ_{can} and ω_{can} . Ours is based on what we call cotangent lifts.

Definition 2.15. *Let Q, Q' be smooth manifolds and $\varphi: Q \rightarrow Q'$ a diffeomorphism. The cotangent lift of φ is defined by*

$$\varphi_!: T^*Q \rightarrow T^*Q', \quad \eta \mapsto (\varphi^{-1})^*\eta.$$

By this, we mean that for any $Y \in TQ'$, we have

$$\langle (\varphi^{-1})^*\eta, Y \rangle_{Q'} = \langle \eta, (D\varphi^{-1})(Y) \rangle_Q,$$

for all $\eta \in T^*Q$. Here $\langle \cdot, \cdot \rangle$ denotes the pairing of covectors with vectors. We will often drop $\langle \cdot, \cdot \rangle$ from the notation and just write $\eta(Z) = \langle \eta, Z \rangle$. Here are some immediate properties of cotangent lifts:

- (1) The cotangent lift preserves fibres, i.e.

$$\pi_{Q'} \circ \varphi_! = \varphi \circ \pi_Q, \tag{12}$$

where $\pi_Q, \pi_{Q'}$ denote the bundle projections of T^*Q and T^*Q' , respectively;

- (2) it acts linearly on fibres;
- (3) it satisfies

$$(\text{id}_Q)_! = \text{id}_{T^*Q}, \quad (\varphi \circ \varphi')_! = \varphi_! \circ \varphi'_!.$$

Now let (U, φ) be a chart $\varphi: U \rightarrow \mathbb{R}^n$ on a subset $U \subset Q$ of the base space. Denote its image by $V = \varphi(U)$. We can view $T^*V = V \times \mathbb{R}^n \subset \mathbb{R}^{2n}$ by trivializing the bundle T^*V by the sections $dq_i: V \rightarrow T^*V$, where we call q_i the standard coordinates on \mathbb{R}^n . In other words, given $q_* \in V$, we identify

$$p_1 dq_1|_{q_*} + \dots + p_n dq_n|_{q_*} \in T_{q_*}^*V \quad \text{with} \quad (q_*, p_1, \dots, p_n) \in V \times \mathbb{R}^n \subset \mathbb{R}^{2n}. \tag{13}$$

Definition 2.16. Now let (U, φ) be a chart of Q as above. The cotangent chart associated to (U, φ) is defined as $\varphi_! : T^*U \rightarrow T^*V$ followed by the above trivialization $T^*V = V \times \mathbb{R}^n$. We denote the resulting map again by

$$\varphi_! : T^*U \rightarrow V \times \mathbb{R}^n, \quad \eta \mapsto (q_1, \dots, q_n, p_1, \dots, p_n).$$

We are now in a position to define the canonical (sometimes called tautological) one-form on T^*Q . Let (U, φ) be a chart and $\varphi_!$ the corresponding cotangent chart. Then we can set

$$(\lambda_{\text{can}})_{(U, \varphi)} = \varphi_!^*(p_1 dq_1 + \dots + p_n dq_n) \in \Omega^1(T^*U). \quad (14)$$

Remark 2.17. Here are two points the reader may be confused about:

- (1) One-forms are smooth sections of the cotangent bundle,

$$\Omega^1(Q) = \Gamma(T^*Q) = \{\alpha : Q \rightarrow T^*Q \mid \pi_Q \circ \alpha = \text{id}_Q\}.$$

However λ_{can} is not a one-form on Q , but on T^*Q ! Therefore it is $\lambda_{\text{can}} : T^*Q \rightarrow T^*T^*Q$ as a map. In fact, this is part of the reason why it is *canonical* or *tautological*: At $\eta \in T^*Q$ it is given by η itself, which can be canonically viewed in $T_\eta^*T^*Q$, see also Proposition 2.27.

- (2) We have slightly abused notation, since by q_i we denote coordinates both on V and on T^*V , compare e.g. (13) and (14). Therefore, we obtain $\pi_Q^* dq_i = dq_i$.

Although the definition (14) is local in Q , it is independent of the choice of (U, φ) and thus yields a globally defined form!

Proposition 2.18. Let (U, φ) and (U', φ') be two charts of Q . Then

$$(\lambda_{\text{can}})_{(U, \varphi)} = (\lambda_{\text{can}})_{(U', \varphi')} \quad \text{on } T^*(U \cap U').$$

Proof. First, we prove the following statement. Let $\chi : V \rightarrow V'$ be diffeomorphism of open sets $V, V' \subset \mathbb{R}^n$. Then the cotangent lift $\chi_! : V \times \mathbb{R}^n \rightarrow V' \times \mathbb{R}^n$ satisfies

$$\chi_!^*(p'_1 dq'_1 + \dots + p'_n dq'_n) = p_1 dq_1 + \dots + p_n dq_n, \quad (15)$$

where (q_i, p_i) and (q'_i, p'_i) denote the coordinates determined by (13) on T^*V and T^*V' , respectively. By Definition 2.15 of the cotangent lift, we can write

$$\chi_!(q, p) = (\chi(q), (D\chi^{-1})^T p).$$

We deduce

$$\chi_!^* dq'_i = d(\chi_i(q_1, \dots, q_n)) = \sum_{j=1}^n \frac{\partial \chi_i}{\partial q_j} dq_j,$$

and

$$p'_i \circ \chi_! = \sum_{k=1}^n p_k \frac{\partial (\chi^{-1})_k}{\partial q'_i}$$

This allows us to compute

$$\begin{aligned}
\chi^* \sum_{i=1}^n p'_i dq'_i &= \sum_{i=1}^n \left(\sum_{k=1}^n p_k \frac{\partial(\chi^{-1})_k}{dq'_i} \right) \left(\sum_{j=1}^n \frac{\partial \chi_i}{\partial q_j} dq_j \right) \\
&= \sum_{i,j,k=1}^n p_k \frac{\partial(\chi^{-1})_k}{dq'_i} \frac{\partial \chi_i}{\partial q_j} dq_j \\
&= \sum_{j,k=1}^n p_k \frac{\partial(\chi^{-1} \circ \chi)_k}{dq_j} dq_j \\
&= \sum_{j=1}^n p_j dq_j.
\end{aligned}$$

Now to prove the claim of the proposition, we set $\chi = \varphi' \circ \varphi^{-1}$ to deduce $(\varphi'_!)^* (\sum_{i=1}^n p'_i dq'_i) = \varphi_!^* (\sum_{i=1}^n p_i dq_i)$ from (15). This proves the claim. \square

Definition 2.19. *The canonical one-form $\lambda_{\text{can}} \in \Omega^1(T^*Q)$ on any cotangent bundle T^*Q is defined by*

$$\lambda_{\text{can}}|_U = (\lambda_{\text{can}})_{(U,\varphi)} = \varphi_!^* (p_1 dq_1 + \dots + p_n dq_n),$$

for any chart (U, φ) on Q . The canonical symplectic form $\omega_{\text{can}} \in \Omega^2(T^*Q)$ is defined as $\omega_{\text{can}} = -d\lambda_{\text{can}}$. In a cotangent chart, it can be seen as

$$\omega_{\text{can}}|_U = \varphi_!^* (dq_1 \wedge dp_1 + \dots + dq_n \wedge dp_n). \quad (16)$$

This form is indeed symplectic. From (16), we can read off that it is closed and non-degenerate. Note that the cotangent charts yield symplectomorphisms of portions $T^*U \subset T^*Q$ of the cotangent bundle with sets of the form $V \times \mathbb{R}^n$ in \mathbb{R}^{2n} equipped with the standard symplectic form $\omega_0 = \sum_i dx_i \wedge dy_i$.

Proposition 2.20. *Let $\psi_! : T^*Q \rightarrow T^*Q'$ be the cotangent lift of any diffeomorphism $\psi : Q \rightarrow Q'$. The canonical one forms are preserved:*

$$\psi_!^* \lambda'_{\text{can}} = \lambda_{\text{can}}.$$

Proof. Let (U, φ) be a chart on Q and (U', φ') be a chart on Q' . Then the map $\chi = \varphi' \circ \psi \circ \varphi^{-1}$ is a diffeomorphism between subsets of \mathbb{R}^n . As in the proof of Proposition 2.18, we find $\chi_!^* (\sum_i p'_i dq'_i) = \sum_i p_i dq_i$. This proves the claim. \square

In classical mechanics, this is extremely helpful. It tells us that we can change coordinate systems on the configuration space Q and still write down the Hamiltonian equation in canonical coordinates.

Another obvious consequence of Proposition 2.20 is that cotangent lifts are symplectomorphisms. Symplectomorphisms preserving a primitive one form (here it is λ_{can}) are sometimes called *exact symplectomorphisms*. We obtain a canonical inclusion

$$\text{Diff}(Q) \hookrightarrow \text{Symp}(T^*Q, \omega_{\text{can}}), \quad \varphi \mapsto \varphi_!. \quad (17)$$

Exercise 2.21 ().* Let $\varphi \in \text{Diff}(Q)$ be a diffeomorphism which is isotopic to the identity through a path in $\text{Diff}(Q)$.

- (1) Show that $\varphi_t \in \text{Ham}(T^*Q, \omega_{\text{can}})$. *Hint:* Pick an explicit path $t \mapsto \varphi_t \in \text{Diff}(Q)$ with $\varphi_0 = \text{id}$ and $\varphi_1 = \varphi$ and show that its cotangent lift $(\varphi_t)_!$ is a Hamiltonian flow (of a time-dependent Hamiltonian vector field).
- (2) Express the Hamiltonian function generating $t \mapsto (\varphi_t)_!$ in terms of the family of vector fields $t \mapsto Y_t \in \Gamma(TQ)$ generating φ_t .

The symplectomorphisms of $(T^*Q, \omega_{\text{can}})$ coming from cotangent lifts are induced by transformations of the base. Let us now consider translations in the fibre.

Definition 2.22. Let $\alpha \in \Omega^1(Q)$ be a one-form. The translation in the fibre along α is defined by

$$t_\alpha: T^*Q \rightarrow T^*Q, \quad \eta \mapsto \eta - \alpha|_{\pi_Q(\eta)}.$$

(The minus here is explained because we want to have a plus in (18).)

Proposition 2.23. Let $\alpha \in \Omega^1(Q)$ be a one-form. Then $t_\alpha^* \lambda_{\text{can}} = \lambda_{\text{can}} - \pi_Q^* \alpha$ and thus

$$t_\alpha^* \omega_{\text{can}} = \omega_{\text{can}} + \pi_Q^* d\alpha. \quad (18)$$

In particular, t_α is a symplectomorphism if and only if $\alpha \in \Omega_{\text{cl}}^1(Q)$.

Proof. Let (U, φ) be a chart of Q . We can write $\alpha = \varphi^*(\sum_i \alpha_i dq_i)$. Then the map $\widehat{t}_\alpha = \varphi_! \circ t_\alpha \circ \varphi_!^{-1}: V \times \mathbb{R}^n \rightarrow V \times \mathbb{R}^n$ is given by

$$\widehat{t}_\alpha(q, p) = (q_1, \dots, q_n, p_1 - \alpha_1, \dots, p_n - \alpha_n).$$

This allows us to compute

$$(\widehat{t}_\alpha)^*(\sum_i p_i dq_i) = \sum_i (p_i - \alpha_i) dq_i = \sum_i p_i dq_i - \pi_Q^*(\sum_i \alpha_i dq_i).$$

In the last equation, we have used that

- (1) the functions α_i depend only on the base coordinates q_i ;
- (2) we have made use of the slight abuse of notation in the sense that $dq_i = \pi_Q^* dq_i$, where $dq_i \in \Omega^1(T^*Q)$ on the LHS and $dq_i \in \Omega^1(Q)$ on the RHS. See Remark 2.17.

Using the definition of \widehat{t}_α , we find

$$\begin{aligned} t_\alpha^* \lambda_{\text{can}} &= t_\alpha^* \varphi_!^*(\sum_i p_i dq_i) \\ &= \varphi_!^*(\sum_i p_i dq_i) - \varphi_!^* \pi_Q^*(\sum_i \alpha_i dq_i) \\ &= \lambda_{\text{can}} - \pi_Q^* \varphi^*(\sum_i \alpha_i dq_i) \\ &= \lambda_{\text{can}} - \pi_Q^* \alpha, \end{aligned}$$

as desired. In the third equality, we have used (12). □

Exercise 2.24 ().* For every $\alpha \in \Omega^1(Q)$, define a unique vector field X_α on T^*Q by

$$\pi_Q^* \alpha = \iota(X_\alpha) \omega_{\text{can}}.$$

Show that its time-one flow $\phi_1^{X_\alpha}$ is exactly t_α . We will crucially use this idea in a later lecture to study the topology of Lagrangian submersions.

Exercise 2.25. When is t_α a Hamiltonian diffeomorphism?

Remark 2.26. Sometimes the *cotangent charts* from Definition 2.16 are dropped from the notation altogether and one just writes

$$\lambda_{\text{can}}|_U = \sum_i p_i dq_i, \quad \omega_{\text{can}}|_U = \sum_i dq_i \wedge dp_i,$$

This makes perfect sense when we interpret the q_i, p_i as functions of the type $T^*U \rightarrow \mathbb{R}$ (obtained from composing the cotangent chart with projection to a coordinate), instead of as coordinates in the image of the cotangent charts. The dq_i are then one forms on T^*U . Physicists call the coordinates q_i, p_i *canonical coordinates*.

On the other hand, there are intrinsic ways of characterizing the canonical one form.

Proposition 2.27. *The canonical one-form λ_{can} can be equivalently defined as follows:*

(1)

$$\lambda_{\text{can}}: TT^*Q \rightarrow \mathbb{R}, \quad \lambda_{\text{can}}(\xi) = (\pi_Q^* \eta)(\xi) = \eta((\pi_Q)_* \xi), \quad (19)$$

where $\eta \in T^*Q$ denotes the footpoint of the vector $\xi \in T_\eta T^*Q$ and $(\pi_Q)_*: TT^*Q \rightarrow TQ$ the differential of the bundle projection $\pi_Q: T^*Q \rightarrow Q$.

(2) λ_{can} is the unique form such that

$$\alpha^* \lambda_{\text{can}} = \alpha, \quad \text{for all } \alpha \in \Omega^1(Q). \quad (20)$$

Note that this equation makes sense, since $\alpha: Q \rightarrow \mathbb{R}$ and thus under its pull-back, λ_{can} is a form on Q .

Proof. Both claims follow from local computations in a cotangent chart. We start with (1). Let (U, φ) be a chart of Q and denote by (q_i, p_i) the coordinates in the image of the cotangent chart $\varphi!$. Recall the identification (13) and write

$$\varphi!(\eta) = (q, p) = \sum_i p_i(\eta) dq_i,$$

and let $\xi' \in T_{(q,p)} T^*V$. This allows us to compute

$$\begin{aligned} \eta((\pi_Q)_* \xi) &= (\varphi!^{-1}(\sum_i p_i(\eta) dq_i))((\pi_Q)_*(\varphi!^{-1})_* (\varphi!)_* \xi) \\ &= (\varphi!^{-1}(\sum_i p_i(\eta) dq_i))(\varphi_*^{-1}(\pi_Q)_*(\varphi!)_* (\xi)) \\ &= \sum_i p_i(\eta) dq_i((\pi_Q)_*(\varphi!)_* \xi) \\ &= \sum_i p_i(\eta) \pi_Q^* dq_i((\varphi!)_* \xi) \\ &= (\sum_i p_i dq_i)((\varphi!)_* \xi) \\ &= \varphi!^*(\sum_i p_i dq_i)(\xi) = \lambda_{\text{can}}(\xi), \end{aligned}$$

where in the second equality we have used $\varphi \circ \pi_Q = \pi_Q \circ \varphi!$, where the third equality follows from the definition of $\varphi!(\eta) = \eta \circ \varphi_*^{-1}$, and where in the fifth equality follows from the abuse of notation $\pi_Q^* dq_i = dq_i$.

Let $\alpha \in \Omega^1(Q)$. We again work in a chart $(U, \varphi: U \rightarrow V)$ and the associated canonical coordinates. Let $\alpha_i \in C^\infty(V)$ such that $\alpha = \varphi^*(\sum_i \alpha_i dq_i)$. Under the identification (13), set

$$\widehat{\alpha}: V \rightarrow T^*V = V \times \mathbb{R}^n, \quad \widehat{\alpha}(q) = (q, \alpha_1(q), \dots, \alpha_n(q)) = \sum_i \alpha_i(q) dq_i$$

Then $\varphi_! \circ \alpha = \widehat{\alpha} \circ \varphi$. Now let $Y \in T_q U$ and compute

$$\begin{aligned} (\alpha^* \lambda_{\text{can}})(Y) &= \varphi_!^*(\sum_i p_i dq_i)(\alpha_* Y) \\ &= (\sum_i p_i dq_i)((\varphi_! \circ \alpha)_* Y) \\ &= (\sum_i p_i dq_i)((\widehat{\alpha} \circ \varphi)_* Y) \\ &= \sum_i \alpha_i dq_i((\pi_Q)_* \widehat{\alpha}_* \varphi_* Y) \\ &= \varphi^*(\sum_i \alpha_i dq_i)(Y) = \alpha(Y). \end{aligned}$$

To show that this property uniquely characterizes λ_{can} , one can use Exercise 2.28, proving that on an open dense subset of TT^*Q we can write any ξ as $\alpha_* Y$. This implies that $\lambda_{\text{can}}(\xi) = \alpha(Y)$ and thus, by continuity, λ_{can} is uniquely determined by this property. \square

Exercise 2.28. Let $\eta \in T_q^*Q$. Prove that the set

$$\{\alpha_* Y \in T_\eta T^*Q \mid Y \in T_q Q, \alpha \in \Omega^1(Q), \alpha|_q = \eta\}$$

is open and dense in $T_\eta T^*Q$. More precisely, it consists of the complement of the vertical distribution $\ker(\pi_Q)_*$ union the zero vector.

2.4. Mechanics on manifolds. Symplectic geometry in cotangent bundles allows us to significantly generalize Hamilton's formalism. Indeed, we can now consider a particle or physical state whose position is constrained to a manifold Q , called *configuration space*.

Definition 2.29. A mechanical datum (Q, g, P) is a triple consisting of a smooth manifold Q , a Riemannian metric g on Q , and a smooth potential function $P \in C^\infty(Q)$.

Classical mechanics on Q is related to symplectic geometry of $(T^*Q, \omega_{\text{can}})$ – this space is called *phase space*. The base coordinate q describes the position of a particle and the fiber component its momentum. Note that although the position is constrained to Q , momenta can take all possible values in the fibre tangent to Q .

Definition 2.30. Let (Q, g, P) be a mechanical datum. The associated mechanical system is given by the Hamiltonian system $(T^*Q, \omega_{\text{can}}, H)$, where we set

$$H = H_{(Q,g,P)}: T^*Q \rightarrow \mathbb{R}, \quad \eta \mapsto \frac{1}{2}g(\eta, \eta) + P(\pi_Q(\eta)). \quad (21)$$

Time evolution of the mechanical system is given by the Hamiltonian flow ϕ^H of $H = H_{(Q,g,P)}$. The first term in (21) is again called *kinetic energy* and the second *potential energy*. Note that we have plugged covectors into the Riemannian metric. We have implicitly used the isomorphism $TQ \rightarrow T^*Q$ induced by $Y \mapsto g(Y, \cdot)$.

Example 2.31. For $Q = \mathbb{R}^n$ and $g = g_{\text{Eucl}}$ the standard Euclidean metric on \mathbb{R}^n , the mechanical system we obtain coincides with the system defined in 2.1 by Hamilton's equation (4) with the same potential function V . This explains the minus sign in $\omega_{\text{can}} = -d\lambda_{\text{can}}$, which is necessary to recover Hamilton's equations with the traditional sign convention.

For any (Q, g, P) mechanical datum and chart $(U, \varphi: U \rightarrow V)$ on Q , we can transport the mechanical datum to obtain a new one

$$(V, g_\varphi = (\varphi^{-1})^*g, P_\varphi = (\varphi^{-1})^*P = P \circ \varphi^{-1}) \quad (22)$$

on the subset V of \mathbb{R}^n . Here the pull-back metric is defined by $(\psi^*g)(Y, Z) = g(\psi_*Y, \psi_*Z)$ for any two vectors $Y, Z \in TQ$.

Proposition 2.32. *For any mechanical datum (Q, g, P) and chart $(U, \varphi: U \rightarrow V)$ on Q , let $(V, g_\varphi, P_\varphi)$ be its push-forward in the chart as defined in (22). Then their respective Hamiltonian systems are conjugate. More precisely:*

$$\varphi! \circ \phi_t^{H(U, g, P)} = \phi_t^{H(V, g_\varphi, P_\varphi)} \circ \varphi! \quad (23)$$

Exercise 2.33 ().* Prove the proposition.

In other words, every mechanical system locally (in Q) reduces to a mechanical system in \mathbb{R}^n and thus a system which obeys Hamilton's equations (3) by which we have started this course! Furthermore, once we have a solution $t \mapsto q(t) \subset V$ in the image of the chart, we can consider

$$\tau(t) = \varphi^{-1}(q(t)) \in Q. \quad (24)$$

to solve the system on Q .

Actually, there is a much more general fact which goes in the same direction:

Proposition 2.34. *Every Hamiltonian system (X, ω, H) on a $2n$ -dimensional symplectic manifold X is locally conjugate to Hamilton's equations in \mathbb{R}^{2n} .*

This proof of the proposition uses the classical *Darboux theorem*.

Theorem 2.35 (Darboux theorem). *Let x be a point in a symplectic $2n$ -manifold (X, ω) . Then there is a neighbourhood U of x and a symplectomorphism $\psi: (U, \omega|_U) \rightarrow \psi(U) \subset (\mathbb{R}^{2n}, \omega_0)$, where $\omega_0 = \sum_{i=1}^n dx_i \wedge dy_i$.*

Exercise 2.36. Use 2.35 to prove 2.34.

At first glance, it may seem that this result makes canonical coordinates and Proposition 2.32 obsolete. Indeed, any Hamiltonian system can locally be written as Hamilton's equations. Note however that Proposition 2.34 is not compatible with the fibration structure of the cotangent bundle. As a consequence, one cannot just project a solution $t \mapsto (q(t), p(t)) \in \mathbb{R}^{2n}$ to the q -coordinate and then transport it to Q as in (24). Consequently, the system obtained by using Darboux's theorem is not mechanical and completely local, whereas Proposition 2.32 is semi-local: it is local in the base coordinates on Q , but extends to the full fibres over it.

2.5. Example: Planar pendulum. Let us now describe the so-called *mathematical* (or *planar*) *pendulum*. Physically speaking, this is an object attached to the end of a rigid arm which moves in the plane $\mathbb{R}^2 = \{(x, y)\}$ equipped with the Euclidean metric $g_{\text{Eucl}} = dx^2 + dy^2$ and which is subjected to a constant gravitational force pointing downwards $F(x, y) = -\partial_y$. Let us fix the mass of the object and the length of the arm to be $= 1$. The potential of the force is $V(x, y) = y$, since we have indeed: $F = -\nabla V$. Note that the Euclidean metric g_{Eucl} is used in the gradient.

The configuration space is not all of \mathbb{R}^2 , but a circle of unit radius around the point in which the rigid arm is fixed – let’s assume the latter is the origin $(0, 0) \in \mathbb{R}^2$. We have

$$Q = \{(x, y) \mid x^2 + y^2 = 1\}.$$

The mechanical system (Q, g, V) is obtained by restricting g_{Eucl} and V to the submanifold Q . Let us now work in a chart of Q . To that end, set

$$\chi: \mathbb{R} \rightarrow Q, \quad \theta \mapsto (\sin \theta, -\cos \theta). \quad (25)$$

It will become clear later why we have chosen this strange parametrization. The map χ is a covering which yields a diffeomorphism $S^1 = \mathbb{R}/(2\pi\mathbb{Z}) \cong Q$. Locally, one can invert it to get charts (U, φ) on Q . Here we will rather work with the globally defined inverse χ of the chart φ than the chart itself. Let us now compute the push-forward datum as in (22) to use Proposition 2.32. For every local inverse φ of χ , we obtain

$$\begin{aligned} g_\varphi &= (\varphi^{-1})^* g_{\text{Eucl}} \\ &= \chi^* g_{\text{Eucl}} \\ &= d(\sin \theta)^2 + d(-\cos \theta)^2 \\ &= (\cos \theta)^2 d\theta^2 + (\sin \theta)^2 d\theta^2 \\ &= d\theta^2. \end{aligned}$$

The induced potential is

$$V_\varphi(\theta) = (V \circ \chi)(\theta) = -\cos \theta.$$

By Proposition 2.32, we can thus solve the Hamiltonian system associated with the mechanical datum

$$(S^1 = \mathbb{R}/(2\pi\mathbb{Z}), g_{S^1} = d\theta^2, V_{S^1} = -\cos \theta). \quad (26)$$

Its phase space is the cotangent bundle T^*S^1 , i.e. the cylinder $\{(\theta, p_\theta) \in S^1 \times \mathbb{R}\}$ equipped with the symplectic form $d\theta \wedge dp_\theta$. The Hamiltonian is

$$H = H_{(S^1, g_{S^1}, V_{S^1})}(\theta, p_\theta) = \frac{p_\theta^2}{2} - \cos \theta. \quad (27)$$

Indeed, the metric g_{S^1} identifies ∂_θ with p_θ and thus $g_{S^1}(p, p) = p_\theta^2$. Recall that autonomous Hamiltonian flows preserve the symplectic form and the Hamiltonian generating them. Therefore, the transformation of T^*S^1 we are looking for preserves the standard area form on the cylinder and the level sets of (27). Let us thus proceed by increasing values of the

energy H . For $H < -1$, the level set is empty. The value $H = -1$ is a critical value, and the level set is the point $(\theta, p_\theta) = (0, 0)$. This is why we have chosen χ the way we did – the stable equilibrium point of the pendulum is then at the origin. The corresponding orbit is the fixed point where the pendulum is at rest. In the range $H \in (-1, 1)$, the values are regular and their level sets are connected closed embedded curves in the cylinder. Thus the system has periodic trajectories which oscillate: At $\theta = 0$ they have momentum $|p_\theta| = \sqrt{2(H + \cos \theta)}$ meaning that all their energy is stored in the kinetic term. They reach their maximal height $H = -\cos \theta$ when $p_\theta = 0$ at which point the energy is purely stored in the potential term. The oscillations can thus be viewed as a continuous trading off of potential energy for kinetic energy (during the fall of the pendulum) and vice-versa (during its ascent). Note that the level sets in the range $H \in (-1, 1)$ consist of one single trajectory. This changes at $H = 1$ which is a singular value. Its level is not a manifold, but rather an *eye* whose ends intersect in the cylinder. This level set consists of three pieces of trajectory. The simplest one is the equilibrium point $\theta = \pi \pmod{2\pi}$ and $p_\theta = 0$. This corresponds to the pendulum standing vertically up. It is unstable and cannot be observed in nature. If $H = 1$ and $p_\theta \neq 0$ there are two possible trajectories distinguished by the sign of p_θ . They are mirror to one another by reflecting the circle with respect to its vertical axis. These trajectories are the only ones which are not periodic. Instead they converge to the point $\theta = \pi$ for time going to $\pm\infty$ but they never reach it. These trajectories cannot be observed in nature either because of friction and material imperfections. The values $H > 1$ are regular and their level sets are disconnected. They consist of two closed embedded curves which are non-contractible in the cylinder, one in the range $p_\theta > 0$ and one in the range $p_\theta < 0$. Physically, these are the trajectories which have enough energy to traverse the north pole $\theta = \pi \pmod{2\pi}$.

3. SYMMETRIES OF HAMILTONIAN SYSTEMS

3.1. Poisson brackets. To motivate the definition of the Poisson-bracket, recall that we have the following chain of inclusions of groups,

$$\text{Ham}(X, \omega) \subset \text{Symp}(X, \omega) \subset \text{Diff}(X).$$

Let us interpret these as infinite-dimensional Lie groups, without worrying too much about what an infinite dimensional manifold is. By differentiating a smooth curve $t \mapsto \psi_t \in \text{Diff}(X)$ satisfying $\psi_0 = \text{id}$, we obtain a smooth vector field on X . Therefore the Lie algebra of $\text{Diff}(X)$ is given by the space of all vector fields. Its Lie bracket is the Lie bracket for vector fields,

$$\text{Lie}(\text{Diff}(X)) = (\Gamma(TX), [\cdot, \cdot]).$$

Since $\text{Ham}(X, \omega)$ is a subgroup, its Lie algebra is a subalgebra of vector fields. In fact, it is the subalgebra of *Hamiltonian vector fields*

$$\Gamma_{\text{Ham}}(TX) = \{X_H \in \Gamma(TX) \mid H \in C^\infty(X)\},$$

equipped with the Lie bracket for vector fields¹. The set of Hamiltonian vector fields is in one-to-one correspondence with the set of smooth functions up to adding a constant function $C_0^\infty(X) = C^\infty(X)/\mathbb{R}$. The induced bracket on functions is the *Poisson bracket*. Here is a very direct definition of it. We will come back the interpretation as a Lie bracket in Proposition 3.5.

Definition 3.1. *Let (X, ω) be a symplectic manifold. The Poisson bracket is defined by*

$$\{-, -\}: C^\infty(X) \times C^\infty(X) \rightarrow C^\infty(X), \quad \{F, G\} = \omega(X_F, X_G).$$

We say that $F, G \in C^\infty(X)$ Poisson-commute if $\{F, G\} = 0$.

By Hamilton's equation, the Poisson bracket can be rewritten as

$$\{F, G\} = dF(X_G) = -dG(X_F), \quad (28)$$

meaning that it measures the rate of change of one of the functions under the Hamiltonian flow of the other. Remarkably, this rate of change is anti-symmetric under exchanging the functions.

Remark 3.2. The Poisson bracket only depends on the functions up to adding constants and is thus well-defined as a map $C_0^\infty(X) \times C_0^\infty(X) \rightarrow C_0^\infty(X)$. Indeed, the Hamiltonian vector field X_F depends only on dF , and not on F itself.

Proposition 3.3. *The Poisson bracket satisfies the following three properties*

- (1) $\{F, G\} = -\{G, F\}$ (anti-symmetry),
- (2) $\{F, GH\} = H\{F, G\} + G\{F, H\}$ (Leibniz rule),
- (3) $\{F, \{G, H\}\} + \{G, \{H, F\}\} + \{H, \{F, G\}\} = 0$ (Jacobi identity),

for all $F, G, H \in C^\infty(X)$.

Exercise 3.4. Prove these properties. *Hint:* For the latter property, compute first

$$\{F, \{G, H\}\} = -\mathcal{L}_{X_F}(\omega(X_G, X_H))$$

and then use Cartan's magic formula, $\mathcal{L}_X = d\iota(X) + \iota(X)d$.

These three properties can be used as axioms to define abstract Poisson brackets. An abstract Poisson bracket does not necessarily come from a symplectic structure, but instead it induces a decomposition of X into a set of symplectic leaves. The study of the geometry one obtains in this way is called *Poisson Geometry*. We will not pursue this any further here and our Poisson bracket always comes from an underlying symplectic structure.

Proposition 3.5. *The Poisson bracket has the following basic properties for all $F, G \in$*

- (1) *If $F, G \in C^\infty(X)$ Poisson-commute, then the functions are constant along each other's Hamiltonian flows,*

$$F \circ \phi_t^G = F, \quad G \circ \phi_t^F = G;$$

¹In particular, the bracket of two Hamiltonian vector fields is again Hamiltonian!

(2)

$$X_{\{F,G\}} = -[X_F, X_G]$$

In particular, the map $F \mapsto X_F$ yields an isomorphism of Lie algebras,

$$(C^\infty(X), \{\cdot, \cdot\}) \rightarrow (\Gamma_{\text{Ham}}(TX), [\cdot, \cdot]);$$

(3) If $F, G \in C^\infty(X)$ Poisson-commute, then their Hamiltonian flows commute, meaning that for all times $t, s \in \mathbb{R}$, we have

$$\phi_t^F \circ \phi_s^G = \phi_s^G \circ \phi_t^F.$$

Proof. To prove (1), apply $\frac{d}{dt}$ to find $dF(X_G)$. The claim follows from (28).

The proof of (2) relies on the Jacobi identity from Proposition 3.3. Recall that any vector field is characterized by how it acts on smooth functions as a derivation. In particular, the Lie bracket can be defined by

$$[X_F, X_G]H = X_F(X_G H) - X_G(X_F H), \quad \text{for all } H \in C^\infty(X).$$

Again, we use that $X_G H = dH(X_G) = \{H, G\}$ to rewrite this expression as $\{\{H, G\}, F\} - \{\{H, F\}, G\} = \{F, \{G, H\}\} + \{G, \{H, F\}\}$. By the Jacobi identity, this is equal to $-\{H, \{F, G\}\} = -X_{\{F,G\}}H$. To summarize, we obtain $[X_F, X_G]H = -X_{\{F,G\}}H$ for all smooth functions H and this proves the claim.

For the proof of (3), recall that if $[Y, Z] = 0$ for any pair of vector fields Y, Z , then their flows $t \mapsto \phi_t^Y$ and $s \mapsto \phi_s^Z$ commute, $\phi_t^Y \circ \phi_s^Z = \phi_s^Z \circ \phi_t^Y$. Let us stress that this is a general fact about *smooth* (and not necessarily Hamiltonian) flows. Now if $\{F, G\} = 0$, we can use (2) to find that $[X_F, X_G] = 0$. Since the Hamiltonian flows of F, G are the flows of the vector fields X_F, X_G meaning, in our notation, that $\phi_t^{X_F} = \phi_t^F$, the claim follows. \square

Exercise 3.6. Find $F, G \in C^\infty(X)$ whose Hamiltonian flows commute, but which do not Poisson-commute, themselves. In other words the converse to point (3) in Proposition 3.5 does not hold.

Let us now discuss how the Poisson bracket is useful in *solving* a Hamiltonian system (X, ω, H) with symmetry. This discussion is *crucial* for later topics in this lecture. Recall that finding an explicit solution is impossible most of the time. Using energy conservation $H \circ \phi_t^H = H$ allows us to look for solutions in the level sets $H^{-1}(h_0)$ of H . This removes one degree of freedom and bring us to dimension $2n - 1$. Although this helps, it is hardly satisfying, except in the case $n = 1$ (as for example in the case of the planar pendulum). Now assume that there is an auxiliary G with $\{H, G\} = 0$.

Then we can use (1) in Proposition 3.5 to find that G preserved under ϕ_t^H , too! In other words, we can now restrict our attention to joint level sets of the form

$$H^{-1}(h_0) \cap G^{-1}(g_0) = \{H = h_0, G = g_0\}.$$

Whenever these are transverse, i.e. when dH, dG are linearly independent (it is obviously of no use to consider $G = 2H$ for example), we lose two degrees of freedom.

We can use (3) in Proposition 3.5 to get a finer idea of what solutions look like in the joint level sets. Indeed, if we have one solution $t \mapsto \phi_t^H(x_0)$, we find a one-parametric family of solutions by transporting it by the Hamiltonian flow of G . Indeed, transporting the initial condition x_0 by ϕ_s^G and then solving the system $t \mapsto \phi_t^H(\phi_s^G(x_0))$ is the same as solving it for x_0 and then applying $\phi_s^G(x_0)$.

Remark 3.7. The function G is sometimes called (*first*) *integral of* (X, ω, H) , in the sense that it helps us *integrate* the system, i.e. solve it.

Noether's theorem can be roughly stated as follows: *For every symmetry of a system, there is an associated preserved quantity.* Using the Poisson bracket formalism, this is almost a tautology. Indeed, the *system* is the Hamiltonian flow ϕ_t^H , the *symmetry* is the Hamiltonian flow ϕ_s^G . The *conserved quantity* is simply the Hamiltonian G itself!

Remark 3.8. A Hamiltonian system (X, ω, H) can have at most $n - 1$ independent symmetries. If it has this maximal amount of symmetry i.e. if there are smooth functions G_2, \dots, G_n such that

$$\{H, G_i\} = \{G_i, G_j\} = 0$$

for all i, j and such that dH, dG_2, \dots, dG_n are linearly independent (at least on an open dense subset of X), then the system is called (*completely*) *integrable*. In that case the candidate space for solutions has dimension $2n - n = n$. The structure of these candidate spaces is elucidated by the so-called *Arnol'd-Liouville theorem*.

3.2. Hamiltonian circle actions. As we have seen, it is extremely useful to find a first integral G of a Hamiltonian system (X, ω, H) because it allows us to restrict our attention to the level set $\{G = g_0\} \subset X$, thus reducing the dimension of the space under consideration by one. Furthermore, the solutions of the system appear in one-dimensional families on that level set. It is thus natural to ask whether one can quotient out by the transformation inducing these one-dimensional families.

Under the additional assumption that the Hamiltonian flow of G generates a circle action, this idea is spectacularly useful. Indeed, in that case the quotient $\{G = g_0\}/S^1$ is itself a symplectic manifold, provided the action on the level set $\{G = g_0\}$ is free. Furthermore, this *symplectic quotient* carries a *residual Hamiltonian system* whose dynamics is intimately related to the dynamics of the original system (X, ω, H) . Note however that the quotient $\{G = g_0\}/S^1$ has dimension $2n - 2$, meaning that we have actually reduced the number of degrees of freedom by two! Before moving on to discussing this *symplectic reduction* procedure, let us have a look at Hamiltonian circle actions.

Definition 3.9. A smooth group action $\psi: S^1 \times X \rightarrow X$ on a symplectic manifold (X, ω) is called Hamiltonian group action if there is an (autonomous) Hamiltonian G such that

$\psi_\theta = \psi(\theta, -)$ is equal to ϕ_t^G for all $t \in \mathbb{R}$ and $\theta = [t] \in S^1 = \mathbb{R}/\mathbb{Z}$. The Hamiltonian G is then called moment map of the action.

In other words, any autonomous Hamiltonian G whose flow satisfies $\phi_1^G = \text{id}$ induces a Hamiltonian S^1 -action on X . We switch between viewing the action as a map $\psi: S^1 \times X \rightarrow X$ and a map $\theta \mapsto \psi_\theta \in \text{Ham}(X, \omega)$.

Here are some examples of Hamiltonian S^1 -actions.

Example 3.10. The system $(\mathbb{R}^2, \omega_0, G(x, y) = \pi(x^2 + y^2))$ defines a Hamiltonian S^1 -action. It is just the rotation of the plane and thus has one fixed point at the origin $(x, y) = (0, 0)$. Note that this is a special case of the harmonic oscillator.

Exercise 3.11. Let $(\mathbb{R}^{2n}, \omega_0 = \sum_i dx_i \wedge dy_i)$ be equipped with the Hamiltonian

$$G_\alpha(x_1, y_1, \dots, x_n, y_n) = \pi\alpha_1(x_1^2 + y_1^2) + \dots + \pi\alpha_n(x_n^2 + y_n^2)$$

for $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$. For which vectors $\alpha \in \mathbb{R}^n$ does this define a Hamiltonian S^1 -action? For which vectors α can G_α be rescaled by some $c \in \mathbb{R}$ to cG_α defining a Hamiltonian S^1 -action?

Example 3.12. Let $S^1 \times \mathbb{R} = \mathbb{R}^2/(x, y) \sim (x+1, y)$ be equipped with the symplectic form $dx \wedge dy$. This space is symplectomorphic to $(T^*S^1, \omega_{\text{can}})$. Then the moment map $G(x, y) = y$ induces a Hamiltonian S^1 -action by translation in the x -direction. Note that it does not have any fixed points.

Example 3.13. Let $S^2 = \{x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3$ be equipped with the scaling ω_{S^2} of the natural area form which satisfies $\int_{S^2} \omega_{S^2} = 2$. Then the height function

$$G: S^2 \rightarrow \mathbb{R}, \quad G(x, y, z) = z$$

defines a Hamiltonian S^1 -action fixing the North and South poles $(0, 0, \pm 1) \in S^2$.

Exercise 3.14 ().* Prove the claims in the previous example. *Hint:* Define a parametrization

$$\chi: S^1 \times [-1, 1] \rightarrow S^2 \setminus \{(0, 0, \pm 1)\},$$

and consider the pull-back system $(S^1 \times [-1, 1], \chi^*\omega_{S^2}, G \circ \chi)$.

Another major source of examples are lifts of smooth actions to cotangent bundles.

Proposition 3.15. *Let $\theta \mapsto \psi_\theta$ be a smooth circle action on a smooth manifold Q . Then its cotangent lift $\theta \mapsto (\psi_\theta)_!$ defines a Hamiltonian circle action on the cotangent bundle $(T^*Q, \omega_{\text{can}})$. We denote this action by $\psi_!: S^1 \times T^*Q \rightarrow T^*Q$.*

Proof. Exercise 2.21 shows that the lift $\theta \mapsto (\psi_!)_\theta = (\psi_\theta)_!$ is a Hamiltonian flow. Since $(\psi_!)_1 = (\psi_1)! = (\text{id}_Q)! = \text{id}_{T^*Q}$, we find that this yields a Hamiltonian S^1 -action. \square

Exercise 3.16. Among Examples 3.10, 3.12 and 3.13, which of them are a lift of a smooth action to a cotangent bundle?

Example 3.17. Let $\mathbb{R}^2 = \{(q_1, q_2)\}$ be equipped with the smooth S^1 -action given by the standard rotation,

$$\psi_\theta(q_1, q_2) = (q_1 \cos 2\pi\theta - q_2 \sin 2\pi\theta, q_2 \cos 2\pi\theta + q_1 \sin 2\pi\theta).$$

Then its cotangent lift $\psi_!$ acts by simultaneous rotation on the coordinate pairs $q = (q_1, q_2)$ and $p = (p_1, p_2)$ of the cotangent bundle $T^*\mathbb{R}^2 = \{(q_1, q_2, p_1, p_2)\}$,

$$(\psi_!)_\theta(q, p) = (\psi_\theta(q), \psi_\theta(p)). \quad (29)$$

One way of seeing this is by noting that, since we are in \mathbb{R}^2 , we can write the cotangent lift in coordinates

$$(\psi_!)_\theta(q, p) = (\psi_\theta(q), (D\psi_\theta^{-1})^T p).$$

But $\psi_\theta: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a linear map! Therefore, we find $D\psi_\theta = \psi_\theta$. Furthermore, since it is a rotation, we have $\psi_\theta^{-1} = \psi_\theta^T$, proving (29). Another approach is computing the Hamiltonian vector field. One can use Exercise 2.21 to find that the moment map generating the lifted action $(\psi_!)_\theta$ is

$$G(q, p) = p(2\pi(q_1\partial_{q_2} - q_2\partial_{q_1})) = p_1q_2 - p_2q_1,$$

since the vector field $Y = 2\pi(q_1\partial_{q_2} - q_2\partial_{q_1})$ generates the rotation on the base $\mathbb{R}^2 = \{(q_1, q_2)\}$. See also Example 2.5. The preserved quantity G of this rotation has a physical interpretation as *angular momentum*.

3.3. Interlude: Geodesics. Let (Q, g) be a Riemannian manifold and ∇ its Levi-Civita connection. A smooth curve $\gamma: [t_0, t_1] \rightarrow Q$ is called *geodesic* if it satisfies

$$\nabla_{\dot{\gamma}} \dot{\gamma} = 0. \quad (30)$$

Such curves are locally length-minimizing. For every tangent vector $Y \in TQ$, there is a unique geodesic γ with $\dot{\gamma}(0) = Y$ which exists at least locally. Existence of such geodesics for all t and Y is called *geodesic completeness*. The Hopf-Rinow theorem tells us that geodesic completeness is equivalent to *metric completeness* of the space Q equipped with the metric induced by g .

Definition 3.18. Let (Q, g) be a geodesically complete Riemannian manifold. The geodesic flow $\Gamma: \mathbb{R} \times TQ \rightarrow TQ$ is defined by

$$\Gamma(t, Y) = \Gamma_t(Y) = \dot{\gamma}(t),$$

where $\gamma: \mathbb{R} \rightarrow Q$ is the unique geodesic with $\dot{\gamma}(0) = Y$.

In the framework of classical mechanics (30) can be interpreted as *the acceleration of γ being zero*. In light of Newton's theorem, this means there is no external force acting on the particle, i.e. that $P = 0$. This is also sometimes called *free particle*. In fact, one can make this rigorous.

Exercise 3.19 (!). Show that the geodesic flow is Hamiltonian in the following sense: Let $g^\# : TQ \rightarrow T^*Q$ denote the isomorphism defined by $g^\#(Y) = g(Y, \cdot)$. Then the geodesic flow is conjugate to the Hamiltonian flow,

$$\phi_t^H = g^\# \circ G_t \circ (g^\#)^{-1},$$

where $H = H_{(Q,g,P=0)}$ denotes the mechanical Hamiltonian with vanishing potential, $P = 0$.

For the sake of this course, we can take this to be the definition of the geodesic flow.

Definition 3.20. Let (Q, g) be a Riemannian manifold. We call $\phi_t^{H_{(Q,g,P=0)}}$ the geodesic flow. The actual geodesics are obtained by projecting to the base space, $\gamma(t) = \pi_Q(\phi_t^H(x_0))$.

Some people call this the *co-geodesic flow* but we will not bother doing this distinction. The Hamiltonian point of view on geodesics is extremely helpful to deal with symmetries of (Q, g) . This is illustrated by the following.

Exercise 3.21 ().* The goal of this exercise is to prove *Clairaut's relation* for the geodesics on surfaces of revolution. For any smooth function $z \mapsto f(z) \in \mathbb{R}_{>0}$, we consider its *surface of revolution* around the z -axis in \mathbb{R}^3 ,

$$\Sigma_f = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = f(z)^2\} \subset \mathbb{R}^3.$$

Convince yourself that the name of this surface is well-deserved. We equip Σ_f with the Riemannian metric g_f induced by the ambient Euclidean metric $g_{\text{Eucl}} = dx^2 + dy^2 + dz^2$. Prove that if γ is a geodesic, then the quantity

$$f(\gamma_z(t)) \cos \alpha(t)$$

is constant in t . Here $\gamma_z(t)$ denotes the z -component of the curve γ and $\alpha(t)$ the angle between $\dot{\gamma}(t)$ and the circle of constant height $\{z = \gamma_z(t)\} \subset \Sigma_f$. Does the converse hold?

3.4. Symplectic reduction. Note that Hamiltonian circle actions preserve the level sets of the moment map generating them. Therefore, we can consider the quotient space of any level set on which S^1 acts *freely*, i.e. for any $x \in G^{-1}(g_0)$, if $\psi_\theta(x) = x$ then $\theta \in S^1$ is the neutral element. Equivalently, the stabilizer of every point in $G^{-1}(g_0)$ is trivial. The quotient turns out to be a symplectic manifold in its own right. This procedure is called *symplectic reduction*.

Theorem 3.22. Let (X, ω) be equipped a moment map $G \in C^\infty(X)$ generating a Hamiltonian S^1 -action which acts freely on the level set $Z = G^{-1}(g_0)$. Then its quotient $X_{g_0} = G^{-1}(g_0)/S^1$ carries a unique symplectic form ω_{g_0} satisfying

$$p^* \omega_{g_0} = \omega|_{TZ}, \tag{31}$$

where $p: Z \rightarrow X_{g_0}$ denotes the natural quotient map.

Proof. Let us first prove that X_{g_0} is a smooth manifold. Firstly, Z is a smooth manifold, since g_0 is a regular value. Indeed, suppose by contradiction that it contains a critical point $x_0 \in G^{-1}(g_0)$. Then $X_G|_{x_0} = 0$ and x_0 is fixed under the Hamiltonian flow of G . This is in contradiction with the action being free. The S^1 -action on the level set is free by hypothesis and automatically proper, since S^1 is compact. Therefore the quotient space is a smooth manifold.

Let us now show that the quotient carries a symplectic form satisfying (31). For all $x \in X_{g_0}$ and $Y, Z \in T_x X_{g_0}$, we can set

$$(\omega_{g_0})_x(Y_1, Y_2) = \omega_{\hat{x}}(\hat{Y}_1, \hat{Y}_2), \quad (32)$$

where $\hat{x} \in X$ is a lift of x and \hat{Y}_1, \hat{Y}_2 are lifts of Y_1, Y_2 in the sense that $p(\hat{x}) = x$ and $p_*(\hat{Y}_i) = Y_i$. Such lifts exist since X_{g_0} is by definition the quotient of Z and thus p is a surjective submersion. If we can prove that ω_{g_0} as defined by (32) is well-defined and symplectic, then we are done. Indeed, the form (32) satisfies (31).

Let $(\hat{x}', \hat{Y}'_1, \hat{Y}'_2)$ be another triple lifting (x, Y_1, Y_2) in the above sense. Let us show that the RHS of (32) yields the same for that triple. First note that there is $\theta_0 \in S^1$ such that $\psi_{\theta_0}(\hat{x}) = \hat{x}'$, but ψ_{θ} acts by Hamiltonian diffeomorphisms and therefore preserves the symplectic form ω ,

$$\omega_{\hat{x}}(\hat{Y}_1, \hat{Y}_2) = (\psi_{\theta_0}^* \omega)_{\hat{x}}(\hat{Y}_1, \hat{Y}_2) = \omega_{\hat{x}'}((\psi_{\theta_0})_* \hat{Y}_1, (\psi_{\theta_0})_* \hat{Y}_2).$$

We thus need to compare $(\psi_{\theta_0})_* \hat{Y}_1$ to \hat{Y}'_1 . To that end, note that

$$p_*((\psi_{\theta_0})_* \hat{Y}_1 - \hat{Y}'_1) = p_*(\hat{Y}_1) - Y_1 = 0,$$

since $p \circ \psi_{\theta} = p$. This proves that $(\psi_{\theta_0})_* \hat{Y}_1 - \hat{Y}'_1$ is tangent to the fibre direction and thus there is λ such that

$$\lambda X_G = (\psi_{\theta_0})_* \hat{Y}_1 - \hat{Y}'_1.$$

It follows that

$$\omega((\psi_{\theta_0})_* \hat{Y}_1, \cdot) = \omega(\hat{Y}'_1 + \lambda X_G, \cdot) = \omega(\hat{Y}'_1, \cdot) + \lambda dG(\cdot),$$

where the latter term vanishes on $TZ = \ker dG$. Hence we obtain a well-defined differential two-form $\omega_{g_0} \in \Omega^2(X_{g_0})$ satisfying (31). This form is closed since $p^* d\omega_{g_0} = (d\omega)|_{TZ} = 0$ by (31) and p is a submersion. Non-degeneracy requires some more thought. Recall that non-degeneracy means that $\omega^\#: TX \rightarrow T^*X$ defined by $\omega^\#(Y) = \iota(Y)\omega$ is an isomorphism. This is the case if and only if

$$\ker \omega^\# = \{Y \in TX \mid \iota(Y)\omega = 0\}$$

is trivial. Although ω is non-degenerate on X , its restriction $\omega|_{TZ}$ is degenerate! Indeed,

$$(\iota(X_G)\omega)_{TZ} = (dG)_{TZ} = 0,$$

since, again, $TZ = \ker dG$. For dimensional reasons (think about this!), we find that X_G is the only degenerate direction, $\ker \omega^\# = \text{span } X_G$. In other words, symplectic reduction

precisely mods out by the only degeneracy of $\omega|_TZ$ and thus the form descends to a non-degenerate one on the quotient.

To formally prove this, we take $Y \in TX_{g_0}$ such that $\omega_{g_0}^\#(Y) = \iota(Y)\omega_{g_0} = 0$ and show that $Y = 0$. Picking \widehat{Y} a lift of Y , we can lift the form,

$$0 = p^*(\iota(Y)\omega_{g_0}) = \omega_{g_0}(Y, p_*(\cdot)) = \omega_{g_0}(p_*\widehat{Y}, p_*(\cdot)) = (\iota(\widehat{Y})\omega)|_{TZ}.$$

Meaning that \widehat{Y} is in the kernel of $\omega|_{TZ}$, and thus a multiple of X_G . Therefore $p_*\widehat{Y} = Y = 0$. \square

Remark 3.23. The projection $p: Z \rightarrow X_{g_0}$ defines a fibre bundle which is given the structure of an S^1 -principal bundle by the action generated by G .

Definition 3.24. We call the so-obtained symplectic manifold (X_{g_0}, ω_{g_0}) the reduced space (at the level $G = g_0$). It is also sometimes called symplectic quotient or Marsden–Weinstein quotient. If the hypotheses of Theorem 3.22 are satisfied, we say that X admits (symplectic) reduction at the level $G = g_0$.

Example 3.25. Let $(\mathbb{R}^{2n+2}, \omega_0)$ be equipped with the Hamiltonian

$$G = \pi(x_0^2 + y_0^2 + \dots + x_n^2 + y_n^2).$$

Setting $z_j = x_j + iy_j$, we can identify $\mathbb{R}^{2n+2} = \mathbb{C}^{n+1}$. In this notation G generates the Hamiltonian S^1 -action $\psi_\theta = \phi_\theta^G$ given by

$$\psi_\theta(z_0, \dots, z_n) = (e^{-2\pi i\theta} z_0, \dots, e^{-2\pi i\theta} z_n).$$

This is sometimes called the *diagonal circle action*. The level set for $g_0 > 0$ is given by the sphere of radius $\sqrt{g_0/\pi}$,

$$G^{-1}(g_0) = S^{2n+1} \left(\sqrt{g_0/\pi} \right).$$

The action is free on the spheres, since, for every point $z = (z_0, \dots, z_n)$ on it, there is at least one $z_j \neq 0$ and ψ_θ acts by rotation on z_j . Therefore, we can perform symplectic reduction. The quotient $X_{g_0} = S^{2n+1}/S^1$ is diffeomorphic to complex projective space $\mathbb{C}P^n$ and the induced symplectic form is called the *Fubini–Study form* $\omega_{\text{FS}} \in \Omega^2(\mathbb{C}P^n)$. Note that there is a freedom in choosing $c > 0$. This freedom corresponds to rescaling the Fubini–Study form, and different authors use different conventions.

Let us now return to the setting of a Hamiltonian system (X, ω, H) which has a first integral G . Assume furthermore that G is the moment map of a Hamiltonian S^1 -action. Whenever G admits symplectic reduction, the reduced space (X_{g_0}, ω_{g_0}) itself carries a Hamiltonian system H_{g_0} induced by H which recovers some of the dynamics of the original system.

Proposition 3.26. Let (X, ω, H) be a Hamiltonian system and G be the moment map of a Hamiltonian S^1 -action on X such that $\{H, G\} = 0$. Assume furthermore that (X, ω) admits

symplectic reduction at the level $G = g_0$. Then the Hamiltonian on the reduced space

$$H_{g_0}: X_{g_0} \rightarrow \mathbb{R}, \quad \text{defined by } H_{g_0} \circ p = H, \quad (33)$$

has the following property

$$p \circ \phi_t^H = \phi_t^{H_{g_0}} \circ p, \quad (34)$$

where $p: G^{-1}(g_0) \rightarrow X_{g_0}$ denotes the natural quotient map.

Exercise 3.27 ().* Prove that (33) determines a well-defined Hamiltonian H_{g_0} and prove the proposition.

Definition 3.28. In the above setup, we call $(X_{g_0}, \omega_{g_0}, H_{g_0})$ the residual system of (X, ω, H) at the level $G = g_0$.

Remark 3.29. All of this holds in the much more general framework of *Hamiltonian group actions*. These are a certain type of group action by a (compact connected) Lie group G_0 on a symplectic manifold generated by a moment map

$$\mu: (X, \omega) \rightarrow \mathfrak{g}^*,$$

taking values in the dual to the Lie algebra \mathfrak{g}^* . This map is G_0 -equivariant with respect to the co-adjoint action of G_0 on \mathfrak{g}^* . Symplectic reduction can still be carried out to yield a symplectic manifold $\mu^{-1}(g_0)/G_0$ whenever $g_0 \in \mathfrak{g}^*$ is invariant under the co-adjoint action and the action on the level set is free.

3.5. Example: Spherical pendulum. The spherical pendulum is a physical system consisting of a rigid arm with one end fixed at a point (the origin, say) of \mathbb{R}^3 and with a weight attached at the other end. It is subjected to a homogeneous gravitational force field. We assume that the arm has no mass and unit length, that there is no friction, and that the weight is concentrated at one point and has unit mass.

In our language, this yields a mechanical system with the following mechanical datum: $Q = S^2 = \{(q_1, q_2, q_3) \mid q_1^2 + q_2^2 + q_3^2 = 1\}$ equipped with the round metric g_{S^2} , i.e. with the restriction of $g_{\text{Eucl}} = dq_1^2 + dq_2^2 + dq_3^2$ to the tangent space of S^2 , and with the potential $P(q_1, q_2, q_3) = q_3$. Recall that the Hamiltonian is given by $H(\eta) = \frac{1}{2}g_{S^2}(V_\eta, V_\eta) + P(\pi_{S^2}(\eta))$, where we have used the metric to define $V_\eta \in TS^2$ as the unique vector such that $\eta = g_{S^2}(V_\eta, \cdot)$. Under the identification

$$T^*S^2 \cong TS^2 = \{(q, p) = (q_1, q_2, q_3, p_1, p_2, p_3) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid \|q\| = 1, p \cdot q = 0\}, \quad (35)$$

we can write

$$H(q, p) = \frac{p_1^2 + p_2^2 + p_3^2}{2} + q_3.$$

Remark 3.30. Writing it out in these coordinates is dangerous – in fact many authors avoid it altogether. The solution to the system is *not* just obtained by looking at Hamilton's equations with respect to the (q_i, p_i) , since this would yield a solution in $(T^*\mathbb{R}^3, \omega_{\text{can}})$, not in the subspace defined by (35).

The spherical pendulum has *rotational symmetry*, i.e. we can rotate S^2 around the q_3 -axis without changing the system. Let us prove this formally. Define the smooth S^1 -action

$$\begin{aligned} \psi: S^1 \times S^2 &\rightarrow S^2, \\ (\theta, (q_1, q_2, q_3)) &\mapsto (q_1 \cos 2\pi\theta - q_2 \sin 2\pi\theta, q_1 \sin 2\pi\theta + q_2 \cos 2\pi\theta, q_3) \end{aligned} \quad (36)$$

The cotangent lift $\psi_!$ of this action yields a Hamiltonian circle action on T^*S^2 , see Proposition 3.15. Since ψ

- (1) acts by isometries on (S^2, g_{S^2}) , and
- (2) preserves the potential P ,

we find that its lift preserves H , i.e. $H \circ (\psi_!)_\theta = H$.

Exercise 3.31. Formalize this argument.

Definition 3.32. We call the Hamiltonian $G \in C^\infty(T^*S^2)$ generating the Hamiltonian circle action $\theta \mapsto (\psi_\theta)_!$ angular momentum.

Since $dH(X_G) = 0$ and we find that

$$\{H, G\} = 0.$$

Thus $(T^*S^2, \omega_{\text{can}}, H)$ has a first integral G .

Remark 3.33. Since we are in dimension four, this is the maximal amount of symmetry one can have, and thus the spherical pendulum is an *integrable system*. Furthermore, this integral generates a Hamiltonian circle action, meaning that we are in a very favorable situation!

The Hamiltonian S^1 -action $\psi_!$ is generated by the function

$$G(q, p) = 2\pi(q_1 p_2 - q_2 p_1).$$

This is angular momentum with respect the rotation around the q_3 -axis.

Exercise 3.34. Prove that G generates $\psi_!$. *Hint:* After you're done, look at your so-called *proof* and find the mistake. Now fix it.

Let us now analyze the dynamics of the pendulum. A fixed point of ϕ_t^H is called *equilibrium* and the equilibria are equal to the critical points of H . Indeed, by Hamilton's equation, these are exactly the points where the vector field X_H vanishes. In the case of a mechanical system (Q, g, P) ,

$$\text{Crit } H = \{(q_0, 0) \in T^*Q \mid q_0 \in \text{Crit } P \subset Q\},$$

i.e. every equilibrium point lies on the zero-section and corresponds to a critical point of the potential P .

Exercise 3.35. Convince yourself of this.

In the case of the spherical pendulum, the potential $P = q_3$ has the north pole $N = (0, 0, 1)$ and the south pole $S = (0, 0, -1)$ as critical points (Recall that we need consider $P|_{S^2}$ as a function on S^2 , not on \mathbb{R}^3). Therefore, the equilibria are $(N, 0), (S, 0) \subset T^*S^2$. These points

correspond to the pendulum being at rest and in a vertical position pointing upwards and downwards respectively.

Since G generates a Hamiltonian S^1 -action, we can perform symplectic reduction as in Theorem 3.22 to obtain a residual Hamiltonian system as in Proposition 3.26. Recall that the system admits reduction on the level $G^{-1}(g_0)$ if the S^1 -action on that level is *free*. A necessary condition is that g_0 be a regular value. Critical points of G are fixed points of the action ψ_1 and those are exactly $(S, 0)$ and $(N, 0)$. The fact that $\text{Crit } H = \text{Crit } G$ is a coincidence! The only *critical value* of G is therefore $g_0 = 0$. It is therefore extremely useful to decompose T^*S^2 into invariant subsets

$$T^*S^2 = \{G = 0\} \sqcup \{G \neq 0\},$$

which we analyze separately. We start with $G = 0$.

Proposition 3.36. *A point $(q_0, p_0) \in T^*S^2$ has vanishing angular momentum, $G(q_0, p_0) = 0$ if and only if its orbit $\gamma(t) = \pi_{S^2}(\phi_t^H(q_0, p_0)) \in S^2$ is contained in a great circle through the North and South Poles, $N = (0, 0, 1)$ and $S = (0, 0, -1)$. Furthermore, the spherical pendulum restricts to a planar pendulum on every such circle.*

Proof. The *if* direction follows either from direct computation or by interpreting G in terms of the vector product,

$$G(q, p) = (p \times q) \cdot e_3. \quad (37)$$

Recall that $p \times q$ is orthogonal to the plane spanned by $p, q \in \mathbb{R}^3$ and thus if p, q are contained in a plane containing N, S then $p \times q$ is orthogonal to e_3 and $G = 0$. The *only if* direction is slightly more involved. Let $(q_0, p_0) \in T^*S^2$ be a point with $G(q_0, p_0) = 0$ and let $\Pi \subset \mathbb{R}^3$ be a plane spanned by q_0, p_0 . If $p_0 = 0$, then we pick such a plane containing the poles N, S . If $p_0 \neq 0$, then the plane is uniquely defined as $\Pi = (p_0 \times q_0)^\perp$. By (37), we find that the condition $G(q_0, p_0) = 0$ shows that this plane contains the North and South Poles. We need to prove that the full orbit is contained in Π .

Because of rotational symmetry, we can assume without loss of generality that $\Pi = \{q_1 = 0\}$. Denote by $R^\Pi: S^2 \rightarrow S^2$ the reflection $(q_1, q_2, q_3) \mapsto (-q_1, q_2, q_3)$ through Π . Its cotangent lift $R_1^\Pi: T^*S^2 \rightarrow T^*S^2$ is given by

$$(q_1, q_2, q_3, p_1, p_2, p_3) \mapsto (-q_1, q_2, q_3, -p_1, p_2, p_3)$$

and preserves H , meaning that $H \circ R_1^\Pi = H$. Note that $R_1^\Pi(q_0, p_0) = (q_0, p_0)$. We claim that the fixed point set

$$\text{Fix } R_1^\Pi = \{(q, p) \in T^*S^2 \mid R_1^\Pi(q, p) = (q, p)\} = \{q_1 = p_1 = 0\}$$

is preserved under the flow ϕ_t^H . Indeed, since R_1^Π is a symplectomorphism, the discussion surrounding (7) shows that

$$(R_1^\Pi)_* X_H = X_{H \circ R_1^\Pi} = X_H.$$

This proves that X_H is tangent to $\text{Fix } R_1^\Pi$ and hence its flow preserves the great circle $\pi_{S^2}(\text{Fix } R_1^\Pi) \subset S^2$. Restricting the metric, the potential and the symplectic form to $\text{Fix } R_1^\Pi$ yields the planar pendulum. Indeed, this can be checked by picking an explicit embedding

$$i: S^1 \times \mathbb{R} \rightarrow T^*S^2, \quad (\theta, p_\theta) \mapsto (\sin \theta, -\cos \theta, p_\theta \cos \theta, p_\theta \sin \theta)$$

which parametrizes $\text{Fix } R_1^\Pi$ and computing the pull-back system $(S^1 \times \mathbb{R}, i^*\omega_{\text{can}}, i^*g_{S^2}, i^*P)$ to find that it agrees with the planar pendulum. \square

Exercise 3.37. The spherical pendulum is not just a collection of planar pendula: Which part of the (*only if* part of the) proof fails when $G \neq 0$?

The set $\{G = 0\}$ consists of an S^1 -family of planar pendula. These intersect in the equilibria at the North and South poles.

Remark 3.38. The circle action generated by G acts on the orbits of the planar pendula in an interesting way. Recall from §2.5 that the planar pendulum has a critical value at $H = 1$ below which its level sets are circles and above which they are a disjoint union of circles. A half-turn ϕ_π^G maps a circle below the critical value to itself, whereas, above the critical value, it interchanges the two connected components of the level set!

Let us move to the invariant subset $\{G \neq 0\}$. Trajectories with $G \neq 0$ do not intersect the poles S, N . Indeed, for every $(q_0, p_0) \in T_S^*S^2 \sqcup T_N^*S^2$ we have $G(q_0, p_0) = 0$. Since ψ_θ acts freely on $S^2 \setminus \{S, N\}$, we deduce that the lift acts freely on $G^{-1}(g_0) \subset T^*S^2 \setminus (T_S^*S^2 \sqcup T_N^*S^2)$. Therefore, for every $g_0 \neq 0$, the system admits symplectic reduction by G . In order to identify the so-obtained quotient spaces, it is convenient to introduce a chart. We choose spherical coordinates

$$\begin{aligned} \varphi: A = S^1 \times (0, 1/2) &\rightarrow S^2 \setminus \{S, N\}, \\ (\theta, \alpha) &\mapsto (\sin 2\pi\alpha \cos 2\pi\theta, \sin 2\pi\alpha \sin 2\pi\theta, \cos 2\pi\alpha), \end{aligned}$$

which yield a diffeomorphism omitting the poles. Note that this diffeomorphism is S^1 -equivariant with respect to the rotation in the first factor on the domain and the action ψ_θ on the target. The cotangent lift yields an exact symplectomorphism

$$\varphi_!: T^*A \rightarrow T^*S^2 \setminus (T_N^*S^2 \cup T_S^*S^2). \quad (38)$$

On $T^*A = T^*(\mathbb{R}/\mathbb{Z} \times (0, 1/2)) = T^*(\mathbb{R}/\mathbb{Z}) \times T^*(0, 1/2)$, we choose the natural coordinates

$$(\theta, \alpha, p_\theta, p_\alpha) \in \mathbb{R}/\mathbb{Z} \times (0, 1/2) \times \mathbb{R} \times \mathbb{R}. \quad (39)$$

Recall that, formally speaking, these are defined by

$$\eta = p_\theta(\eta)d\theta_{(\theta(\eta), \alpha(\eta))} + p_\alpha(\eta)d\alpha_{(\theta(\eta), \alpha(\eta))}.$$

for all $\eta \in T^*A$. In the cotangent chart, the Hamiltonians $H_\varphi = H \circ \varphi_!$, $G_\varphi = G \circ \varphi_!$ look as follows

$$H_\varphi(\theta, \alpha, p_\theta, p_\alpha) = \frac{1}{8\pi^2} \left(\frac{p_\theta^2}{(\sin 2\pi\alpha)^2} + p_\alpha^2 \right) + \cos 2\pi\alpha,$$

$$G_\varphi(\theta, \alpha, p_\theta, p_\alpha) = 2\pi p_\theta.$$

Exercise 3.39 ()*. Prove this. *Hint*: For G , you should not compute anything. For H , use the discussion surrounding Proposition 2.32. You'll need some familiarity with Riemannian metrics for that.

Remark 3.40. The Hamiltonian system $(T^*A, \omega_{\text{can}}, H_\varphi)$ is *not complete*!! Recall that this means that the flow $\phi_t^{H_\varphi}$ may not be defined for all times, depending on which point we start with. Indeed, solutions of $(T^*S^2, \omega_{\text{can}}, H)$ intersect at least one of the fibre $T_N^*S^2, T_S^*S^2$ if and only if $G = 0$. Since this is an equivalence, the system becomes complete when we restrict our attention to $\{G_\varphi \neq 0\} = T^*A \setminus \{p_\theta = 0\}$. Since this set is equal to $\varphi_!^{-1}(G \neq 0)$, it is exactly the subset we are interested in, here!

We are now ready to perform symplectic reduction with respect to G .

Proposition 3.41. *The residual Hamiltonian system of $(T^*S^2, \omega_{\text{can}}, H)$ at the level $G = 2\pi g_0 \neq 0$ is given by*

$$(T^*S^2)_{g_0} \cong T^*(0, 1/2) = (0, 1/2) \times \mathbb{R} = \{(\alpha, p_\alpha)\},$$

$$(\omega_{\text{can}})_{g_0} = \omega_{\text{can}}^{T^*(0, 1/2)} = d\alpha \wedge dp_\alpha,$$

$$H_{g_0} = \frac{p_\alpha^2}{8\pi^2} + \frac{g_0^2}{8\pi^2(\sin 2\pi\alpha)^2} + \cos 2\pi\alpha.$$

Proof. Since $G \neq 0$, the level set $G = 2\pi g_0 \neq 0$ is completely contained in the image of $\varphi_!$ and thus we can work with the residual system of $(T^*A, \omega_{\text{can}}, H_\varphi)$ at the level $G_\varphi = 2\pi g_0$, instead. In the coordinates (39), the level set is

$$G_\varphi^{-1}(2\pi g_0) = \mathbb{R}/\mathbb{Z} \times (0, 1/2) \times \{g_0\} \times \mathbb{R},$$

and the action on it is the obvious S^1 -action on the first factor. Therefore symplectic reduction simply eliminates the first factor in $T^*A = T^*(\mathbb{R}/\mathbb{Z}) \times T^*(0, 1/2)$. Since the symplectic form also splits, $\omega_{\text{can}} = d\theta \wedge dp_\theta + d\alpha \wedge dp_\alpha$ we find that the reduced space is $(T^*(0, 1/2), d\alpha \wedge dp_\alpha)$. The residual Hamiltonian H_φ is simply the restriction to the level set, which yields the expression we have claimed. Note that its well-definedness on the symplectic quotient is reflected in the fact that H_φ does not depend on θ . \square

The Hamiltonian H_{g_0} is again mechanical, and its potential is

$$P_{\text{eff}, g_0}(\alpha) = \frac{g_0^2}{8\pi^2(\sin 2\pi\alpha)^2} + \cos 2\pi\alpha. \quad (40)$$

This is called the *effective potential*, because a part of the kinetic energy term of H has been absorbed into the potential.

Definition 3.42. *A point $x \in G^{-1}(g_0)$ is called relative equilibrium if it maps to a critical point of the residual Hamiltonian H_{g_0} , i.e. if $p(x) \in \text{Crit } H_{g_0}$.*

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