ISSG 26 – EXERCISES

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Abstract.

Exercises which are difficult or require specialized knowledge are marked by *(!)*.

1. March 17th

Exercise 1.1 (Ex. 2.21). Let $\varphi \in \mathrm{Diff}(Q)$ be a diffeomorphism which is isotopic to the identity through a path in $\mathrm{Diff}(Q)$.

(1) Show that $\varphi_! \in \mathrm{Ham}(T^*Q, \omega_{\mathrm{can}})$. *Hint:* Pick an explicit path $t \mapsto \varphi_t \in \mathrm{Diff}(Q)$ with $\varphi_0 = \mathrm{id}$ and $\varphi_1 = \varphi$ and show that its cotangent lift $(\varphi_t)_!$ is a Hamiltonian flow (of a time-dependent Hamiltonian vector field).
(2) Express the Hamiltonian function generating $t \mapsto (\varphi_t)_!$ in terms of the family of vector fields $t \mapsto Y_t \in \Gamma(TQ)$ generating φ_t.

Exercise 1.2 (Ex. 2.24). For every $\alpha \in \Omega^1(Q)$, define a unique vector field X_α on T^*Q by

$$\pi_Q^*\alpha = \iota(X_\alpha)\omega_{\mathrm{can}}.$$

Show that its time-one flow $\phi_1^{X_\alpha}$ is exactly t_α. We will crucially use this idea in a later lecture to study the topology of Lagrangian submersions.

Proposition 1.3. *For any mechanical datum (Q, g, P) and chart $(U, \varphi\colon U \to V)$ on Q, let $(V, g_\varphi, P_\varphi)$ be its push-forward in the chart as defined in (**??**). Then their respective Hamiltonian systems are* conjugate*. More precisely:*

$$\varphi_! \circ \phi_t^{H_{(U,g,P)}} = \phi_t^{H_{(V,g_\varphi,P_\varphi)}} \circ \varphi_!. \tag{1}$$

Exercise 1.4 (Ex. 2.33). Prove the proposition.

Example 1.5. Let $S^2 = \{x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3$ be equipped with the scaling ω_{S^2} of the natural area form which satisfies $\int_{S^2} \omega_{S^2} = 2$. Then the height function

$$G\colon S^2 \to \mathbb{R}, \quad G(x, y, z) = z$$

defines a Hamiltonian S^1-action fixing the North and South poles $(0, 0, \pm 1) \in S^2$.

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Exercise 1.6 (Ex. 3.14). Prove the claims in the previous example. *Hint:* Define a parametrization

$$\chi\colon S^1 \times [-1,1] \to S^2 \setminus \{(0,0,\pm 1)\},$$

and consider the pull-back system $(S^1 \times [-1,1], \chi^*\omega_{S^2}, G \circ \chi)$.

Exercise 1.7 (! Ex. 3.19). Show that the geodesic flow is Hamiltonian in the following sense: Let $g^\# \colon TQ \to T^*Q$ denote the isomorphism defined by $g^\#(Y) = g(Y, \cdot)$. Then the geodesic flow is conjugate to the Hamiltonian flow,

$$\phi_t^H = g^\# \circ G_t \circ (g^\#)^{-1},$$

where $H = H_{(Q,g,P=0)}$ denotes the mechanical Hamiltonian with vanishing potential, $P = 0$.

Exercise 1.8 (3.21). The goal of this exercise is to prove *Clairaut's relation* for the geodesics on surfaces of revolution. For any smooth function $z \mapsto f(z) \in \mathbb{R}_{>0}$, we consider its *surface of revolution* around the z-axis in $\mathbb{R}^3$,

$$\Sigma_f = \{(x,y,z) \in \mathbb{R}^3 \,|\, x^2 + y^2 = f(z)^2\} \subset \mathbb{R}^3.$$

Convince yourself that the name of this surface is well-deserved. We equip Σ_f with the Riemannian metric g_f induced by the ambient Euclidean metric $g_{\mathrm{Eucl}} = dx^2 + dy^2 + dz^2$. Prove that if γ is a geodesic, then the quantity

$$f(\gamma_z(t)) \cos \alpha(t)$$

is constant in t. Here $\gamma_z(t)$ denotes the z-component of the curve γ and $\alpha(t)$ the angle between $\dot\gamma(t)$ and the circle of constant height $\{z = \gamma_z(t)\} \subset \Sigma_f$. Does the converse hold?

Proposition 1.9. *Let (X, ω, H) be a Hamiltonian system and G be the moment map of a Hamiltonian S^1-action on X such that $\{H, G\} = 0$. Assume furthermore that (X, ω) admits symplectic reduction at the level $G = g_0$. Then the Hamiltonian on the reduced space*

$$H_{g_0} \colon X_{g_0} \to \mathbb{R}, \qquad \textit{defined by } H_{g_0} \circ p = H, \tag{2}$$

has the following property

$$p \circ \phi_t^H = \phi_t^{H_{g_0}} \circ p, \tag{3}$$

where $p\colon G^{-1}(g_0) \to X_{g_0}$ denotes the natural quotient map.

Exercise 1.10 (Ex. 3.27). Prove that (2) determines a well-defined Hamiltonian H_{g_0} and prove the proposition.

References

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